Solutions to Problems in Midterm 1:

1. Any reversible engine operating on Carnot cycles between 2 reservoirs with the same T_H & T_L has an efficiency:

$$e = 1 - \frac{Q_L}{Q_H} = 1 - \frac{T_L}{T_H} .$$

$$e = \frac{W}{Q_H} .$$

Since the Carnot cycle for E1 & E_2 also have the same max & min pressures, same max & min volumes, then they also have the same Q_H:

$\rightarrow$ $Q_H = NkT_H \ln \frac{V_B}{V_A}$ (see graph for Carnot cycle)

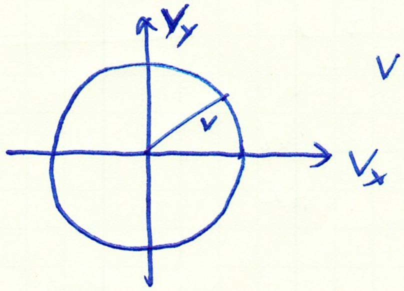
$W = e\, Q_H$; $e_{E_1} = e_{E_2}$

$$\therefore \boxed{W_{E_1} = W_{E_2}}$$

2. a). In 2D:

$$B(\vec{v}) = f_x(v_x) f_y(v_y) = \frac{1}{\left(\frac{2\pi kT}{m}\right)^{2/2}} e^{-\frac{1}{2}m(v_x^2+v_y^2)/kT}$$

In the velocity configuration space:



$V = \sqrt{v_x^2 + v_y^2}$

Any point along a circle of radius $V = \sqrt{v_x^2+v_y^2}$ has the same distribution $B(\vec{v})$

Area of this circle is: $A(v) = 2\pi v \, dv$.

$$\therefore f_{2D}(v) = 2\pi v \frac{1}{2\pi kT/m} e^{-\frac{1}{2}mv^2/kT}$$

I'll leave it up to you whether you normalize to N or 1.

b). Equipartition theorem:

$\frac{1}{2}kT$ per degree of freedom for kinetic energy

so in 2D:

$$\left\langle \frac{1}{2}mv^2 \right\rangle = 2 \cdot \frac{1}{2}kT = kT$$

$$\langle v^2 \rangle = \frac{2kT}{m}$$

$$v_{rms} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{2kT}{m}}$$

2 c). When the laser light is abruptly turned off, there is no heat transfer to the gas $\Rightarrow Q = 0$

no work done either $\Rightarrow W = 0$.

thus $\Delta U = 0$.

So after the atoms thermolize in 3D

$$U_{3D} = U_{2D}$$

$$\frac{3}{2} N k T_{3D} = \frac{2}{2} N k T_{2D}$$

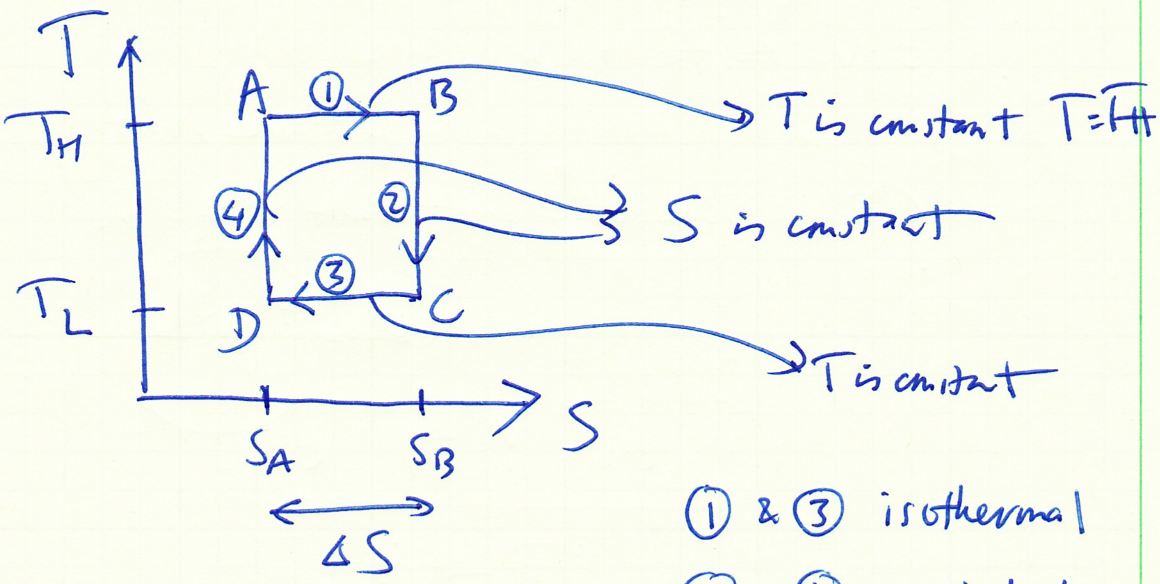
$$\boxed{T_{3D} = \frac{2}{3} T_{2D}}$$

$\therefore$ the temperature of the 3D gas after thermalization is $\frac{2}{3} T_{2D}$, the original temperature in 2D.

d). $$f_{3D}(v) = 4\pi v^2 \frac{1}{(2\pi kT/m)^{3/2}} e^{-\frac{1}{2}mv^2/T_{3D}}.$$

H

3. a/. T-S diagram of Carnot cycle:

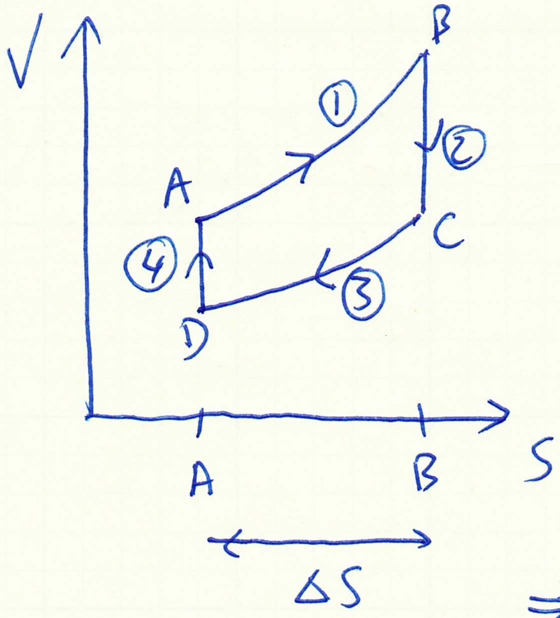


① & ③ isothermal

② & ④ adiabatic.

b/.

V-S diagram of the same Carnot cycle:



② & ④ S = constant.

① : $V \propto e^{S/Nk}$

③ : $V \propto e^{S/Nk}$ $\Rightarrow$

$\Rightarrow$ from ΔS for isothermal process 4 ideal gas

$$\Delta S = \frac{NkT}{T}\int_{V_1}^{V_2}\frac{dV}{V} = Nk\ln\frac{V_2}{V_1}$$

4. $U = \sigma V T^4$

a/. $C_V = \left(\frac{\partial U}{\partial T}\right)_V = 4\sigma V T^3$

b/. From 1st Law:

$$dU = TdS - pdV$$

at constant V, $dV = 0$, $dU = C_V dT$

$$C_V dT = TdS$$

$$\int_0^S dS = \int_0^T \frac{C_V}{T} dT$$

$$S(T) - \underset{0}{\underset{\shortparallel}{S(0)}} = \int_0^T \frac{4\sigma V T^3}{T} dT$$

$$\boxed{S(T,V) = \frac{4\sigma V T^3}{3} = \frac{1}{3} C_V}$$

c/. Adiabatic Expansion: $\Delta S = 0$

$$S(T,V) = \frac{4\sigma V T^3}{3} = \text{constant.}$$

$$\boxed{VT^3 = \text{const.}}$$

4 d). $pV = (\gamma - 1) U$

$= (\gamma - 1) \sigma V T^4$

$p = (\gamma - 1) \sigma T^4$.

Combining $p = (\gamma - 1)\sigma T^4$ with 4c) $VT^3 = \text{const}$.

$\Rightarrow \quad p^{3/4} V = \text{const}$.

$$\boxed{pV^{4/3} = \text{const.}} \Longleftarrow$$

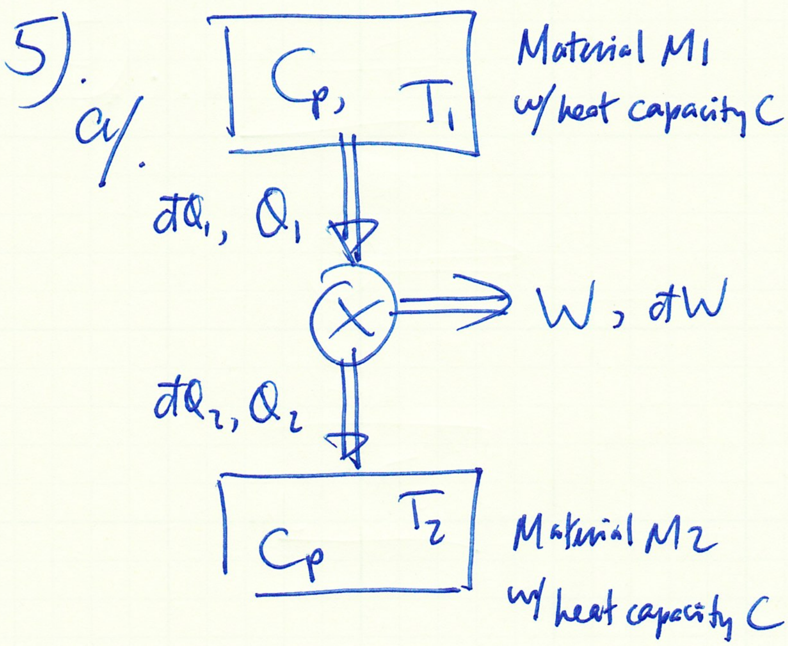
e). K_s = compressibility at constant entropy.

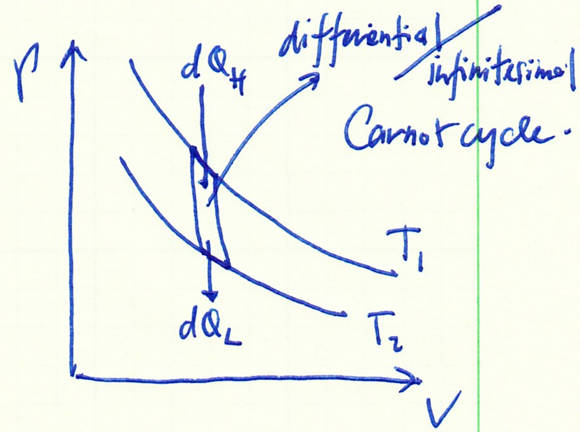
$$= \frac{-1}{V}\left(\frac{\partial V}{\partial p}\right)_S$$

$$V = \frac{\text{const}}{p^{3/4}}$$

$$\boxed{K_s = \frac{3}{4}\frac{1}{p}}$$

5). a/.





To denote the variable of temperature of M1 : t_1

~ M2 : t_2

For each differential/infinitesimal Carnot cycle :

$$\frac{|đq_h|}{t_1} = \frac{|đq_l|}{t_2} \longleftarrow \text{from } \frac{đq_h}{t_1} + \frac{đq_l}{t_2} = 0$$

For each $đq_h$ & $đq_l$, temperature of M_1 & M_2 change by :

(1) $dt_1 = -đq_h / C_p$

(2) $dt_2 = đq_l / C_p$; where C is the heat capacity of both M_1 & M_2.

(3) $đq_l = \frac{t_2}{t_1} đq_h$

(2) & (3) $\Rightarrow$ $dt_2 = \frac{1}{C_p}\frac{t_2}{t_1} đq_h$ (4)

(1) & (4) $\Rightarrow$

$$dt_2 = \frac{1}{\cancel{C_p}}\frac{t_2}{t_1}(-\cancel{C_p}dt_1)$$

$$\frac{dt_2}{t_2} = -\frac{dt_1}{t_1}$$

$$\int_{T_2}^{T_f}\frac{dt_2}{t_2} = -\int_{T_1}^{T_f}\frac{dt_1}{t_1}$$

Integrating over many cycles until both reservoirs reach T_f

$$\ln\frac{T_f}{T_2} = -\ln\frac{T_f}{T_1} = \ln\frac{T_1}{T_f}$$

$$\therefore \quad \frac{T_f}{T_2} = \frac{T_1}{T_f} \Rightarrow \boxed{T_f = \sqrt{T_1 T_2}}$$

b).

$$|Q_{1\,total}| = |C_p(T_f - T_1)| = C_p(T_1 - T_f)$$

$$|Q_{2\,total}| = |C_p(T_f - T_2)| = C_p(T_f - T_2)$$

From 1st Law:

$$W_{tot} = |Q_{1\,total}| - |Q_{2\,total}|$$

$$= C_p(T_1 - T_f) - C_p(T_f - T_2)$$

$$\boxed{W_{tot} = C_p(T_1 + T_2 - 2T_f)}$$

5c).
$$\Delta S_{M1} = C_p \ln \frac{T_f}{T_1} < 0$$

$$\Delta S_{M2} = C_p \ln \frac{T_f}{T_2} > 0$$

$$= -\Delta S_{M1} \text{ since } \frac{T_f}{T_1} = \frac{T_2}{T_f}.$$

$$\Delta S_{Engine} = 0 \longrightarrow \text{Carnot cycle}$$

+

$$\boxed{\Delta S_{total} = 0}$$

the whole process is reversible!

(just like in a standard textbook / canonical Carnot engine/cycle).