SOLUTIONS

TO

MATH 54 MIDTERM 2

Nov 13 2014 12:40-2:00pm

Your Name Student ID	

Section number and leader	

Do not turn this page until you are instructed to do so.

Show all your work in this exam booklet. There are pages with extra space at the end. No material other than simple writing utensils may be used. *In the event of an emergency or fire alarm leave your exam (closed) on your seat and meet with your GSI outside.*
If you need to use the restroom, leave your exam with your GSI while out of the room.

Your grade is determined from all of the following 5 problems. Some extra credit problems are interspersed and can make up for up to 5 missed points within the same problem. However, only complete answers earn this credit, so check your other work before attempting these.

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[5] **1. (a)** (Without reasoning) extend $p_1(t) = t^2 - 1, p_2(t) = 3t^3$ to a basis of $\mathbb{P}_3$.

$p_3(t) = t \quad , \; p_4(t) = t^2$

(or $1 \, , t$ or ....)

[8] **(b)** Let $\mathbf{u}$ and $\mathbf{v}$ be linearly independent vectors in a vector space V, and let $\mathbf{w}$ be another vector in V that does not lie in $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$. Show that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent.

$c_1 \underline{u} + c_2 \underline{v} + c_3 \underline{w} = \underline{0}$ implies $c_1 = c_2 = c_3 = 0$ because:

<u>cases</u>

- $c_3 \neq 0 \quad \Rightarrow \quad \underline{w} = -\frac{c_1}{c_3} \underline{u} - \frac{c_2}{c_3} \underline{v}$ in $\mathrm{Span}\{\underline{u}, \underline{v}\}$

  contradicts asssumption

- $c_3 = 0 \quad \Rightarrow \quad c_1 \underline{u} + c_2 \underline{v} = \underline{0}$

  $\Rightarrow \quad c_1 = c_2 = 0$ by lin. indep. of $\underline{u}, \underline{v}$

[7] **(c)** Decide whether t, e^t, e^{t^2} are linearly independent. State explicitly any theorems that you are using.
(extra credit) Prove a theorem that relates Wronskians of general functions with linear independence.

Thm: Wronskian $(t=0) \neq 0 \implies$ lin. indep.

$$W(t) = \det \begin{bmatrix} t & e^t & e^{t^2} \\ 1 & e^t & 2te^{t^2} \\ 0 & e^t & 2e^{t^2} + (2t)^2 e^{t^2} \end{bmatrix}$$

$$\Rightarrow W(0) = \det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} = -1 \det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = -(2-1) = -1 \neq 0$$

$\Downarrow$

lin. indep.

(extra) : "Wronskian at $t_0 \neq 0 \implies$ functions linearly independent"

$\iff$ "$W(t) = 0$ for all $t \impliedby$ functions linearly dependent"

$\Updownarrow$ $\Updownarrow$

$$\det \begin{bmatrix} y_1(t) & \cdots & y_n(t) \\ \vdots & & \vdots \\ y_1^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{bmatrix} = 0 \text{ for all } t \qquad c_1 y_1 + \cdots + c_n y_n = 0 \quad \text{for some } \underline{c} \neq \underline{0}$$

$\Downarrow$ by derivaties $\Uparrow$ evident (by def$^{\underline{n}}$ from 1st row)

$\Uparrow$

$$\begin{bmatrix} y_1(t) & \cdots & y_n(t) \\ \vdots & & \vdots \\ y_1^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{bmatrix} \underline{c} = \underline{0} \text{ for all } t \text{ and some } \underline{c} \neq \underline{0} \iff \left\{ \begin{matrix} c_1 y_1(t) + \cdots + c_n y_n(t) = 0 \\ c_1 y_1'(t) + \cdots + c_n y_n'(t) = 0 \\ \\ c_1 y_1^{(n-1)}(t) + \cdots + c_n y_n^{(n-1)}(t) = 0 \end{matrix} \right\} \text{ for all } t \text{ and some } \underline{c} \neq \underline{0}$$

[6] **2. (a)** Let H be a nonempty collection of matrices. What other properties does H have to satisfy in order to be a subspace of $M_{2\times 2}$?

- each matrix in H is 2×2
- A, B in $H \Rightarrow A+B$ in H
- A in H, c scalar $\Rightarrow cA$ in H

(or A, B in H, c scalar $\Downarrow$ $A+cB$ in H)

[7] **(b)** Let V be the vector space of solutions to $y'' + 4y = 0$. Find the matrix of the linear transformation $T : V \to \mathbb{R}^2, y \mapsto \begin{bmatrix} y(0) \\ y'(\pi) \end{bmatrix}$ relative to the basis $\mathcal{B} = \{\cos 2t, \sin 2t\}$ for V and the standard basis of $\mathbb{R}^2$. Then use this matrix to decide whether T is onto or one-to-one.

(extra credit) Explain what this says about existence and uniqueness of solutions to $y'' + 4y = 0$.

$$T[\cos 2t] = \begin{bmatrix} \cos 0 \\ -2\sin 2\pi \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad T[\sin 2t] = \begin{bmatrix} \sin 0 \\ 2\cos 2\pi \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$\Rightarrow T$ represented by $\underline{\underline{A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}}$

Since A invertible and $T[y] = A(P_{\mathcal{B}}[y])$ for coordinate isomorphism $P_{\mathcal{B}} : V \to \mathbb{R}^2$, T is also invertible ($T^{-1}(\underline{x}) = P_{\mathcal{B}}^{-1}(A^{-1}\underline{x})$), thus onto and one-to-one.

(extra) $\begin{cases} y'' + 4y = 0 \\ y(0) = v_0 \\ y'(\pi) = v_1 \end{cases}$ has a unique solution for any v_0, v_1

(...though this is <u>not</u> guaranteed by the existence & uniqueness thm)

[7] **(c)** Let $T : \mathbb{R}^2 \to \mathbb{R}^2, \mathbf{x} \mapsto A\mathbf{x}$ be the linear transformation given by the matrix $A = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$. Find a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ of $\mathbb{R}^2$ so that the linear transformation T in $\mathcal{B}$-coordinates is $[T]_{\mathcal{B}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Know: If $A(\underline{b}_1 + i\,\underline{b}_2) = (a - ib)(\underline{b}_1 + i\underline{b}_2)$ then $P_{\mathcal{B}} = [\underline{b}_1, \underline{b}_2]$ transforms A into standard rotation form $P_{\mathcal{B}}^{-1} A P_{\mathcal{B}} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

So solve $A(\underline{b}_1 + i\underline{b}_2) = -i\,(\underline{b}_1 + i\underline{b}_2)$:

$$\text{Nul}\begin{bmatrix} -1+i & -2 \\ 1 & 1+i \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} -1-i \\ 1 \end{bmatrix}\right\}$$

$$\begin{bmatrix} -1-i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + i\begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

(read off from 2^{nd} row; check 1^{st} row: $(-1+i)(-1-i) - 2\cdot 1 = (-1)^2 - i^2 - 2 = 1 + 1 - 2 = 0$)

$$\Rightarrow \underline{b}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \underline{b}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

CHECK:

$$\underbrace{\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}^{-1}}_{\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}} \underbrace{\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}}_{\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

[6] **3. (a)** (Without reasoning) determine the dimensions of the kernel and range of a linear transformation T that is represented by the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$.

pivots

dim kernel T (= # columns without pivot) = $\underline{\underline{1}}$

dim range T (= # rows with pivot) = $\underline{\underline{2}}$

[7] **(b)** Let $T : V \to W$ and $S : W \to V$ be linear transformations so that $S(T(v)) = v$ for all v in V. Give a proof to one of the claims that T is onto resp. that T is one-to-one.

(extra credit) Add an extra assumption on the dimensions of V, W and show that T is an isomorphism.

T is one-to-one:

$T(v) = 0 \implies v = S(T(v)) = S(0) = 0$

(extra) $\dim V = \dim W = n$ finite

$\implies$ T represented by $n \times n$ matrix A with $A\underline{x} = \underline{0} \implies \underline{x} = \underline{0}$

$\implies$ A invertible $\implies$ T invertible

$\Updownarrow$

one-to-one and onto

[7] **(c)** State the definition of two matrices A and B being similar. Then show how each of $\det(A^2)$, $Nul(A)$, and $\dim Nul(A)$ changes under a similarity transformation.

A similar to B : $A = P^{-1}BP$ for some P

- $\underline{\underline{\det A^2}} = \det(P^{-1}BPP^{-1}BP) = (\det P)^{-1} \det B \det P = \underline{\underline{\det B}}$

- $\underline{\underline{Nul(A)}} = \{\underline{x} \mid A\underline{x} = \underline{0}\} = \{\underline{x} \mid P^{-1}BP\underline{x} = \underline{0}\}$

  $= \{\underline{x} \mid BP\underline{x} = \underline{0}\}$ since $P\underline{z} = \underline{0} \Leftrightarrow \underline{z} = \underline{0}$

  substitute $\underline{y} = P\underline{x}$

  $= P^{-1}\{\underline{y} \mid B\underline{y} = \underline{0}\}$

  $= \underline{\underline{P^{-1} Nul(B)}}$

- $\underline{\underline{\dim Nul(A) = \dim Nul(B)}}$ since $P : Nul(B) \to Nul(A)$ isomorphism

[10] **4. (a)** Find a basis for the kernel of the differential operator $L = \frac{1}{6}(D^2+5)^2(D^2-2D+2)$.

(extra credit) Calculate $L[e^t]$.

aux. polynomial: $\frac{1}{6}(r^2+5)^2(r^2-2r+2)$

roots: $r = \pm i\sqrt{5}$, $r = 1 \pm \sqrt{1-2} = 1 \pm i$

multiplicity: 2 each , 1 each

$\Rightarrow$ basis of $\{L[y]=0\}$: $\cos\sqrt{5}t$, $\sin\sqrt{5}t$, $t\cos\sqrt{5}t$, $t\sin\sqrt{5}t$,
$e^t\cos t$, $e^t\sin t$

(extra) $L[e^{rt}] = \frac{1}{6}(r^2+5)^2(r^2-2r+2)$

$r=1$: $L[e^t] = \frac{1}{6}(1+5)^2(1-2+2) = 6$

[10] **(b)** $y'' + 2y' + 4y = \sqrt{3}e^{-t}\cos\sqrt{3}t$ is a resonant problem because $-1+\sqrt{3}i$ is a root of the auxiliary equation. Find a solution $y(t)$ using the method of a complex Ansatz with varying parameter.
Hints: You may use any other method for up to 6 points, but beware of cumbersome algebra.
For the given method, you'll need the following steps:
- *Write $\sqrt{3}e^{-t}\cos\sqrt{3}t$ as real part of a complex function $g(t)$.*
- *Use the Ansatz $z(t) = c(t)e^{(-1+i\sqrt{3})t}$ to rewrite $z'' + 2z' + 4z = g(t)$ into an ODE for $c(t)$.*

Hint: Due to resonance, the terms with $c(t)$ should cancel.
- *Find a particular (complex) solution of the ODE for $c(t)$. Hint: Try a simple polynomial.*
- *Plug in and deduce a real solution.*

- $g(t) = \sqrt{3}e^{(-1+\sqrt{3}i)t}$
- $z = c\,e^{(-1+\sqrt{3}i)t} \Rightarrow z' = \left(c' + (-1+\sqrt{3}i)c\right)e^{(-1+\sqrt{3}i)t}$

$$\Rightarrow z'' = \left(c'' + 2(-1+\sqrt{3}i)c' + (-1+\sqrt{3}i)^2 c\right)e^{(-1+\sqrt{3}i)t}$$

ODE: $z'' + 2z' + 4z = g$

$$\Updownarrow$$

$$\left(c\left(\underbrace{(-1+\sqrt{3}i)^2 + 2(-1+\sqrt{3}i) + 4}_{=0}\right) + c'\left(\underbrace{2(-1+\sqrt{3}i) + 2}_{=2\sqrt{3}i}\right) + c''\right)e^{(-1+\sqrt{3}i)t} = \sqrt{3}e^{(-1+\sqrt{3}i)t}$$

$$\Updownarrow$$

- $\boxed{2\sqrt{3}i\,c' + c'' = \sqrt{3}}$
- particular solution $c(t) = \frac{1}{2i}t$ $\left(\Rightarrow c' = \frac{1}{2i},\ c'' = 0\right)$
- $y(t) = \mathrm{Re}\left(\frac{1}{2i}t\,e^{(-1+\sqrt{3}i)t}\right) = \mathrm{Re}\left(-\frac{1}{2}t\,i\,e^{-t}(\cos\sqrt{3}t + i\sin\sqrt{3}t)\right)$

$$= \underline{\underline{\frac{1}{2}te^{-t}\sin\sqrt{3}t}}$$

[6] **5. (a)** Rewrite the ODE $y^{(3)} - 3y'' + 5y' + 7y = \ln(2+t)$ for $y : (-2, \infty) \to \mathbb{R}$ into an equivalent system for a vector function $\mathbf{x} : (-2, \infty) \to \mathbb{R}^n$.

(extra credit) Explain why no component of a solution $\mathbf{x}$ can be a polynomial.

$$\underline{x}(t) = \begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \end{bmatrix} \rightsquigarrow \underline{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -5 & 3 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ \ln(2+t) \end{bmatrix}$$

(extra) if y is a polynomial then $y^{(3)} - 3y'' + 5y' + 7y$ = polynomial

cannot solve $= \ln(2+t)$

if y' or y'' is a polynomial then (by integration) so is y

[7] **(b)** Find the general solution of $\mathbf{x}' = A\mathbf{x}$ for $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix}^{-1}$.

Hint: You can read off eigenvalues and eigenvectors.

$$A\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad A\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = 5\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \quad A\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} = -1\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

gen. sol$^{\underline{n}}$ $\underline{x}(t) = c_1 e^{5t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} + c_3 e^{-t}\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$

[7] **(c)** Find the solution of

$$x' = 2x + 5y \qquad x(0) = -1$$
$$y' = 5x + 2y \qquad y(0) = 1$$

Hint: To accelerate work you can use the fact that $\underbrace{\begin{bmatrix} 2 & 5 \\ 5 & 2 \end{bmatrix}}_{A} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ -3 \end{bmatrix}}_{= -3 \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$.

$\Rightarrow \underline{x}(t) = e^{-3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is a solution of $\underline{x}' = A\underline{x}$

$\underline{x}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow$ it's <u>the</u> solution with $\underline{x}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\Rightarrow \boxed{x(t) = -e^{-3t}, \quad y(t) = e^{-3t}}$