Problem 1

a)

The flux is

$$\Phi_B = \int \vec{B} \cdot d\vec{A} = Bh(x_2 - x_1)$$

Where x_1 and x_2 are the positions of the two rods. Using Faraday's law

$$|\mathcal{E}| = \frac{d\Phi_B}{dt} = Bh(v_0 - v_1)$$

The current this EMF makes will be such that the change in flux is resisted (Lenz's law). This means that the magnetic field due to the induced current should point outward and thus the current should go counterclockwise. The magnitude of this current is

$$I = \frac{\mathcal{E}}{2R} = \frac{Bh(v_0 - v_1)}{2R}$$

The force on the right rod is then

$$\vec{F} = I\vec{l} \times \vec{B} = \frac{B^2h^2(v_0 - v_1)}{2R}(-\hat{y} \times -\hat{z}) = \frac{B^2h^2(v_0 - v_1)}{2R}\hat{x}$$

This is equal to $M\frac{dv_1}{dt}$. Thus,

$$\frac{dv_1}{dt} = -\frac{B^2h^2}{2RM}v_1 + \frac{B^2h^2v_0}{2RM}$$

The solution to this is on the equation sheet.

$$v_1 = v_0(1 - e^{-\frac{B^2h^2}{2RM}t})$$

b)

After a long time $t \gg \frac{B^2h^2}{2RM}$, the exponential term becomes small and $v_1 \to v_0$. This is expected as if the two rods are traveling at the same speed, the flux through the area contained by them will no longer change, and thus the time derivative of the flux will be zero and there will be no induced EMF.

Problem 2

2(a)

Using the right hand grip rule, we see that the straight and semicircular segments all produce fields at P that point into the page. Call this the $-\hat{z}$ direction.

2(b)

Applying Biot-Savart to the semicircular segment we have:

$$\mathbf{B_c} = \frac{\mu_0 I}{4\pi}\int \frac{\mathbf{dl}\times\hat{\mathbf{r}}}{r^2} = -\frac{\mu_0 I}{4\pi}\int_0^{\pi}\frac{d d\theta}{d^2}\hat{\mathbf{z}} = -\frac{\mu_0 I}{4d}\hat{\mathbf{z}}. \tag{1}$$

Using the formula for the magnitude of B for an infinite wire, $B = \frac{\mu_0 I}{2\pi R}$, the two straight segments each contribute half of this. This can be argued either by symmetry or by doing the Biot-Savart integral from $x = -\infty$ to $x = 0$. Hence the total field at P is:

$$\mathbf{B} = -\frac{\mu_0 I}{4\pi d}\hat{\mathbf{z}} - \frac{\mu_0 I}{4\pi d}\hat{\mathbf{z}} - \frac{\mu_0 I}{4d}\hat{\mathbf{z}} = -\frac{\mu_0 I}{4d}\left(1+\frac{2}{\pi}\right)\hat{\mathbf{z}} \tag{2}$$

2(c)

Since $\mathbf{v}$ is parallel to $\mathbf{B}$, $\mathbf{v}\times\mathbf{B} = 0$, so the Lorentz force exerted on the electron is zero: $\mathbf{F} = q\mathbf{v}\times\mathbf{B} = 0$.

Problem 3

$\mathbf{E} = 0$ inside conductors. The net charge of the overall system is $+Q_0$.

3(a)

(i) Pick a Gaussian surface that lies entirely within the thick conducting shell (it doesn't even have to be spherical). Since $\mathbf{E} = 0$, we have:

$$\oint \mathbf{E}\cdot d\mathbf{A} = 0 \Rightarrow Q_{enc} = 0. \tag{3}$$

The inner shell has positive charge $2Q_0$, so the inner surface of the thick shell must have $-2Q_0$.

(ii) The outer shell has overall net charge $-Q_0$, which means there will be a charge of $+Q_0$ residing on its outer surface if there is $-2Q_0$ on its inner surface.

3(b)

Spherical symmetry and Gauss's Law give us $|\mathbf{E}| = kQ_{enc}/r^2$, so:

$$\mathbf{E} = \begin{cases} 0 & \text{for } r < R_1 \\ \frac{2Q_0}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} & \text{for } R_1 \le r < R_2 \\ 0 & \text{for } R_2 \le r < R_3 \\ \frac{Q_0}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} & \text{for } R_3 \le r \end{cases} \tag{4}$$

Using $V = -\int \mathbf{E} \cdot d\mathbf{l}$, we integrate inwards along a radial line from $r = \infty$ (so that $V(\infty) = 0$ is our reference point).

$$V(R_3) = -\int_{\infty}^{R_3} \frac{Q_0}{4\pi\epsilon_0 r^2} dr = \frac{Q_0}{4\pi\epsilon_0 R_3} \tag{5}$$

From R_3 to R_2, $\mathbf{E}$ is zero so there will be no contribution to V in the line integral, hence $V(R_3) = V(R_2)$ (just like the midterm 2 problem).

For R_1:

$$V(R_1) = -\int_{\infty}^{R_1} \mathbf{E} \cdot d\mathbf{l} = V(R_3) - \int_{R_2}^{R_1} \frac{2Q_0}{4\pi\epsilon_0 r^2} dr = \frac{Q_0}{4\pi\epsilon_0}\left(\frac{2}{R_1} - \frac{2}{R_2} + \frac{1}{R_3}\right) \tag{6}$$

As before, since $\mathbf{E} = 0$ for $r < R_1$ inside the thin shell, there is no change in potential and $V(R_1) = V(R_2)$.

3(c)

Connecting the two spheres with a conducting wire means their potentials will be the same, and charges will move to accomodate this. Looking at the calculation we just did in part (b), the potential change between R_2 and R_1 is given by $\Delta V = -\int_{R_2}^{R_1} \mathbf{E} \cdot d\mathbf{l}$. If $V(R_1) = V(R_2)$, then $\Delta V = 0$ and the only way this is possible is if $\mathbf{E} = 0$ in the region between the two spheres $R_1 \le r \le R_2$. So, by Gauss's law, there is no charge on the inner thin sphere, and all the charge has moved to the outer surface of the thick conducting sphere! There will therefore be a charge $+Q_0$ on the outer surface of the outer sphere, and no charge on any of the other surfaces, and the potential of the system will be $V = \frac{Q_0}{4\pi\epsilon_0 R_3}$. This makes sense intuitively because by connecting the spheres we have essentially made them into one big conductor and charge always moves to the outer surface of a conductor.

Problem 4

4(a)

In order to do this problem one needs to recall the mechanical analogs of capacitors and inductors. One easy way to remember this is to look at the expressions for energy in the two systems: KE $= \frac{1}{2}mv^2$, PE $= \frac{1}{2}kx^2$, $U_C = \frac{q^2}{2C}$, and $U_L = \frac{1}{2}LI^2$. Now $v = \frac{dx}{dt}$, and in a circuit $I = \frac{dq}{dt}$, which would suggest the correspondence:

$$x \leftrightarrow q, \quad v \leftrightarrow I, \quad m \leftrightarrow L, \quad k \leftrightarrow C^{-1}. \tag{7}$$

Our circuit will therefore have two capacitors and one inductor. Recall that the spring constants for two springs in series add in the same way resistors do in parallel:

$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2} \implies k_e = \frac{k_1 k_2}{k_1 + k_2}. \tag{8}$$

So the equation of motion for our mass spring system will be:

$$\ddot{x} = -\frac{k_e}{m}x \implies \omega^2 = \frac{1}{m\left(\frac{1}{k_1} + \frac{1}{k_2}\right)} \tag{9}$$

Under our mapping (7) this becomes:

$$\omega^2 = \frac{1}{L\left(C_1 + C_2\right)}. \tag{10}$$

Our circuit will therefore have an equivalent capacitance of $C_1 + C_2$, so it will be an inductor L in series with two capacitors C_1 and C_2 in parallel.

4(b)

The differential equation for the charge will be just as in the mass-spring system:

$$\ddot{q} = -\frac{1}{L\left(C_1 + C_2\right)}q. \tag{11}$$

The solution of this differential equation is of the form: $Q = Q_0 \cos(\omega t + \delta)$ where Q_0 and δ represent the amplitude and phase of the oscillation which will be fixed by initial conditions, and ω is as given above.

Problem 5

Let's define a coordinate system such that the plates are at $x = \pm d/2$. The problem is asking us to determine when the configuration with the sphere hanging between the two plates ($x = 0$) ceases to be stable, so we should recognize this as a balancing of forces problem. The sphere will feel two types of force - gravitational and electromagnetic. We'll calculate these two forces as a function of displacement from equilibrium, and see when the net force ceases to be restoring.

Electromagnetic. It is important to realize that the sphere need not be neutral. Connecting it to ground sets its electric potential to zero, and charge will flow (via the wire) onto or off of the sphere until this is the case. The potential at x due to the plates is

$$V_p(x) = V_0 \left(\frac{2}{d} x \right). \tag{1}$$

The potential at the surface of the sphere due to the charge is

$$V_Q(x) = \frac{Q}{4\pi\epsilon_0 R}. \tag{2}$$

The total potential at the surface of the sphere is the sum of these two contributions. Setting this sum equal to zero gives us the charge on the sphere:

$$\frac{Q}{4\pi\epsilon_0 R} + V_0 \left(\frac{2}{d} x \right) = 0 \longrightarrow Q = -\frac{8\pi\epsilon_0 V_0 R}{d} x. \tag{3}$$

Now we can find the electric force on the sphere.

$$\vec{F}_e = Q\vec{E} = -Q\vec{\nabla}V = -Q\frac{2V_0}{d}\hat{x} = \frac{16\pi\epsilon_0 V_0^2 R}{d^2} x\hat{x}. \tag{4}$$

We've used the fact that the electric field must be perpendicular to the planes by symmetry (they can be approximated as infinite) and must be uniform by Gauss's law (consider a Gaussian cube with two faces parallel to the planes) to establish that $dV/dx = \Delta V/\Delta x$.

Gravitational. The total gravitational force on the sphere is $\vec{F} = -Mg\hat{z}$. If the wire is at an angle θ from the vertical, the component of the force perpendicular to the wire is then

$$\vec{F}_{g,\perp} = -Mg\sin\theta = -\frac{Mg}{L} x\hat{\theta}. \tag{5}$$

For small angles $\hat{\theta} \approx \hat{x}$, and we can write

$$\vec{F}_g = -\frac{Mg}{L} x\hat{x}. \tag{6}$$

Total. The net force on the sphere is just the sum of these two forces:

$$\vec{F}_{\text{tot}} = \vec{F}_g + \vec{F}_e = \left(\frac{16\pi\epsilon_0 V_0^2 R}{d^2} - \frac{Mg}{L} \right) x\hat{x}. \tag{7}$$

Notice that this is a Hooke's Law force. The equilibrium at $x = 0$ will be unstable, and any small perturbation will result in the sphere moving towards one of the plates, when the coefficient is positive. Rearranging this inequality gives the corresponding condition on V_0:

$$|V_0| > \sqrt{\frac{Mgd^2}{16\pi\epsilon_0 LR}}. \tag{8}$$

Problem 6

A. We can think about the resistor as being composed of spherical shell resistors of area $4\pi r^2$, length dr, and resistivity $\rho(r)$. These resistors have resistance $dR = \rho(r)L/A = \rho(r)dr/4\pi r^2$. They are connected in series, so must all have the same current I flowing through them. Then the drop in voltage across one of these resistors is $dV = IdR = I\rho(r)dr/4\pi r^2$. Because electric field is the negative gradient of potential, the fact that the field doesn't depend on r implies that dV/dr is also independent of r. This means that $dV/dr \propto \rho(r)/r^2$ is independent of r, so that $\rho(r) \propto r^2$, and we conclude that $s = 2$.

B. The total resistance of the spherical resistor is just the sum of the resistances of all the shells, because resistors add in series:

$$R = \int dR = \int_a^b \frac{\rho(r)dr}{4\pi r^2} = \int_a^b \frac{\rho_0}{4\pi a^2} = \frac{\rho_0}{4\pi}\frac{b-a}{a^2}. \tag{9}$$

Then the current is given by Ohm's law:

$$I = \frac{V}{R} = \frac{4\pi a^2 V}{\rho_0(b-a)}. \tag{10}$$

C. The circuit heats the water by dissipating power at the resistor. The dissipated power is

$$P = IV = \frac{4\pi a^2 V^2}{\rho_0(b-a)}. \tag{11}$$

Therefore the energy dissipated during a time interval of length t is

$$\Delta E = Pt = \frac{4\pi a^2 V^2}{\rho_0(b-a)}t. \tag{12}$$

Change in temperature is related to energy transfer by the specific heat:

$$\Delta E = Mc_w(T - T_0) \longrightarrow T = \frac{\Delta E}{Mc_w} + T_0 = \left(\frac{4\pi a^2 V^2}{Mc_w\rho_0(b-a)}\right)t + T_0. \tag{13}$$

Problem 7

A. The efficiency of a heat engine is defined to be $e = W/Q_{\text{in}}$, the amount of work output (benefit) per heat input (cost). For the real heat engine, there is not enough information given to be any more specific. For a Carnot engine, we can use the given formula (which you should be able to derive):

$$e = 1 - \frac{T_c}{T_h} = 1 - \frac{T_L}{2T_L} = \frac{1}{2}. \tag{14}$$

B. Differential entropy change is given by $dS = dQ/T$, where dS is entropy gained by the system and Q is heat put into the system. Therefore, we can write

$$\Delta S = \oint_{\text{cycle}} dS = \oint_{\text{cycle}} \frac{dQ}{T}. \tag{15}$$

This is the best we can do without knowing more about the details of the engine.

C. The change in entropy of the universe during a single cycle of the Carnot engine is given by

$$\Delta S = -\frac{Q_{\text{in}}}{T_H}{}_{\frac{Q_{\text{out}}}{T_C}} = \frac{1}{T_L}\left(-\frac{Q_{\text{in}}}{2} + Q_{\text{out}}\right) = \frac{1}{T_L}\left(-\frac{Q_{\text{in}}}{2} + Q_{\text{in}} - W\right) \tag{16}$$

$$= \frac{1}{T_L}\left(\frac{Q_{\text{in}}}{2} - eQ_{\text{in}}\right) = \frac{1}{T_L}\left(\frac{Q_{\text{in}}}{2} - \frac{Q_{\text{in}}}{2}\right) = 0. \tag{17}$$

We've used $dS = dQ/T$, $e = W/Q_{\text{in}}$, and $Q_{\text{in}} = W + Q_{\text{out}}$.