EECS 20N: Structure and Interpretation of Signals and Systems MIDTERM 2
Department of Electrical Engineering and Computer Sciences 9 March 2009
UNIVERSITY OF CALIFORNIA BERKELEY

LAST Name Kauzil FIRST Name Auntie

Lab Time Always

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except two double-sided $8.5" \times 11"$ sheet of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 8.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the eight numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

You may or may not find the following information useful:

Convolution Sum: The input-output relation for a discrete-time LTI system is described by the convolution sum:

$$y(n) = \sum_{k=-\infty}^{+\infty} h(k)x(n-k) = \sum_{\ell=-\infty}^{+\infty} x(\ell)h(n-\ell),$$

where x is the input and y is the output.

Convolution Integral: The input-output relation for a continuous-time LTI system is described by the convolution integral:

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{+\infty} x(\lambda)\, h(t-\lambda)\, d\lambda,$$

where x is the input and y is the output.

Frequency response of a discrete-time LTI system: If the system's impulse response is h, then its frequency response (assuming it exists), is defined as follows:

$$\forall \omega \in \mathbb{R}, \quad H(\omega) = \sum_{n=-\infty}^{+\infty} h(n)\, e^{-i\omega n}.$$

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$$\forall \omega \in \mathbb{R}, \quad H(\omega) = \int_{-\infty}^{+\infty} h(t)\, e^{-i\omega t}\, dt.$$

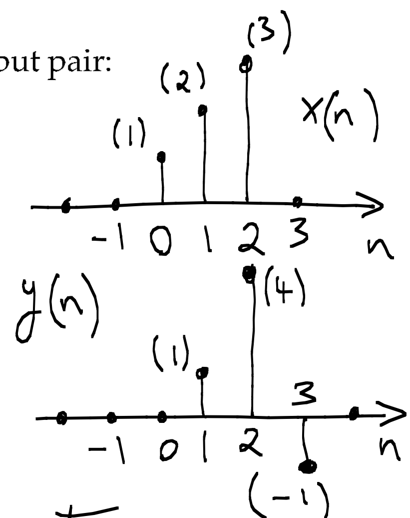
MT2.1 (45 Points) A system is said to be *anticausal* if it never uses past input values to determine the current or future output values. More concretely, a discrete-time system is anticausal if for any $N \in \mathbb{Z}$ for which there exists a pair of input signals x_1 and x_2 such that $x_1(n) = x_2(n)$ for all $n \geq N$, the corresponding output signal values are equal: $y_1(n) = y_2(n)$ for all $n \geq N$.

Each of the following parts refers to a generic discrete-time system H. Information that we give about the system H in one part may *not* be carried over to another part.

For each part, you must explain your answer succinctly, but clearly and convincingly.

(a) Suppose the system H is *linear*, and has the following input-output pair:

$$\begin{aligned} x(n) &= \delta(n) + 2\delta(n-1) + 3\delta(n-2) \\ y(n) &= \delta(n-1) + 4\delta(n-2) - \delta(n-3). \end{aligned}$$



Select the strongest true assertion from the list below.

(i) The system must be anticausal.

(ii) The system could be anticausal, but does not have to be.

$\boxed{\text{(iii) The system cannot be anticausal.}}$

The system is linear $\Longrightarrow$ Zero input produces zero output: $\hat{x}(n) = 0 \;\; \forall n \longrightarrow \hat{y}(n) = 0 \;\; \forall n$

We note that $x(n) = \hat{x}(n) = 0 \quad \forall n \geq 3$, but

$-1 = y(3) \neq \hat{y}(3) = 0$.

We can use this kind of reasoning to show (more generally) that if the input to a linear, anticausal system is left-sided (i.e., $x(n) = 0 \;\; \forall n \geq N$), then the corresponding output must be left-sided with $y(n) = 0 \;\; \forall n \geq N$)

(b) Suppose the system H is *time invariant*, and has the same input-output pair specified in part (a). Remember, you can *not* assume that the system is linear in this part. Select the strongest true assertion from the list below.

(i) The system must be anticausal.

(ii) The system could be anticausal, but does not have to be.

(iii) The system cannot be anticausal.

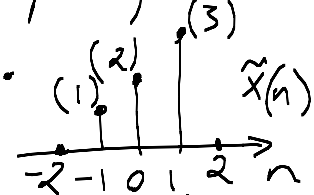
Let $\tilde{x}(n) = x(n+1)$ (advance x by one sample).

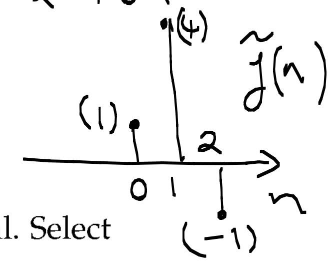
Since H is TI, we know $\tilde{y}(n) = y(n+1)$.

Note that $\tilde{x}(n) = x(n)$ $\forall n \geq 3$, but

$0 = \tilde{y}(3) \neq y(3) = -1$.

We can generalize this result in a manner similar to part (a)





(c) Suppose the system H is LTI (with impulse response h) and anticausal. Select the strongest true assertion from the list below.

(i) $h(n) = 0$ for all $n < 0$.

(ii) $h(n) = 0$ for all $n \leq 0$.

(iii) $h(n) = 0$ for all $n > 0$.

(iv) $h(n) = 0$ for all $n \geq 0$.

(v) $h(n) = 0$ for all $|n| \leq N$, where N is a positive integer.

(vi) $h(n) = 0$ for all $|n| \geq N$, where N is a positive integer.

does not pose a problem

previous value of x

$$y(n) = \sum_{m=-\infty}^{\infty} h(m)x(n-m) = \cdots + h(-1)x(n+1) + h(0)x(n) + \underbrace{h(1)x(n-1) + \cdots}_{\text{We can't have these terms}}$$

$$h(1) = h(2) = \cdots \underset{\text{must be}}{=} 0$$

MT2.2 (15 Points) Consider a continuous-time BIBO stable system H. Let $h(t)$ and $H(\omega)$ denote the impulse response and frequency response values of the system, respectively.

Prove that the magnitude response of the system is bounded; that is, prove

$$|H(\omega)| \quad \infty, \quad \forall \omega.$$

$$|H(\omega)| = \left| \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt \right| \leq \int_{-\infty}^{\infty} \left| h(t) e^{-i\omega t} \right| dt = \int_{-\infty}^{\infty} |h(t)| dt \Bigg\} \Rightarrow$$

We know H is BIBO stable $\Rightarrow \int |h(t)| dt < \infty$

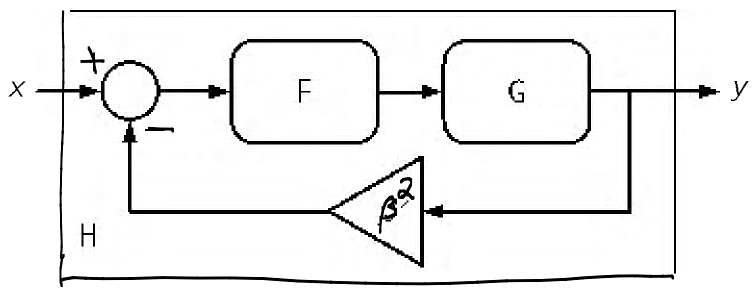
$$|H(\omega)| < \infty \quad \forall \omega$$

You may use the blank space below for scratch work. Nothing written below this line on this page will be considered in evaluating your work.

MT2.3 (45 Points) Consider a BIBO stable continuous-time LTI system F. Let $f(t)$ and $F(\omega)$ denote the impulse response and frequency response values of the system, respectively. Additionally, assume that $f(t) \in \mathbb{R}$ for all t (this is important).

Let a related LTI system G be defined such that its impulse response g is the time-reversed version of f. That is, $g(t) = f(-t)$ for all t.

The two systems are placed in a feedback configuration as shown in the figure below.



Assume $\beta \in \mathbb{R}$, so $\beta^2 > 0$. Here are some potentially Useful Facts:

- If $q(t) = e^{-\lambda t}u(t)$, where $\text{Re}(\lambda) > 0$, then $Q(\omega) = \frac{1}{\lambda + i\omega}$.
- If $r(t) = e^{-\lambda|t|}$, where $\text{Re}(\lambda) > 0$, then $R(\omega) = \frac{2\lambda}{\lambda^2+\omega^2}$.
- $\int_A^B e^{\mu t}, dt = \frac{e^{\mu B} - e^{\mu A}}{\mu}$.

(a) Show that the frequency response $H(\omega)$ of the feedback interconnection is given by

$$H(\omega) = \frac{|F(\omega)|^2}{1 + \beta^2 |F(\omega)|^2}.$$

Hint: First show that $G(\omega) = F^*(\omega)$.

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt = \int_{-\infty}^{\infty} f(-t) e^{-i\omega t} dt = \int_{\infty}^{-\infty} f(\tau) e^{i\omega\tau}(-d\tau) = \int_{-\infty}^{\infty} f(\tau) e^{i\omega\tau} d\tau = F(-\omega)$$

let $\tau = -t$

But $f(t) \in \mathbb{R} \Rightarrow F^*(\omega) = \int_{-\infty}^{\infty} f^*(t) e^{i\omega t} dt = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt = F(-\omega)$. So $G(\omega) = F^*(\omega)$

Now use Black's formula: $H(\omega) = \dfrac{\text{Forward Gain}}{1 - \text{Loop Gain}} = \dfrac{F(\omega)F^*(\omega)}{1+\beta^2 F(\omega)F^*(\omega)} \Rightarrow$

$$H(\omega) = \frac{|F(\omega)|^2}{1+\beta^2|F(\omega)|^2}$$

(b) Suppose the impulse response of the system F is described as follows:

$$f(t) = e^{-t}u(t).$$

Determine $H(\omega)$ and $h(t)$, the frequency response and the impulse response of the composite feedback system H, respectively. Explain how the feedback configuration can be used to increase the bandwidth of F.

$$f(t) = e^{-t}u(t) \Rightarrow F(\omega) = \frac{1}{1+i\omega} \Rightarrow |F(\omega)|^2 = \frac{1}{1+\omega^2} \Rightarrow$$

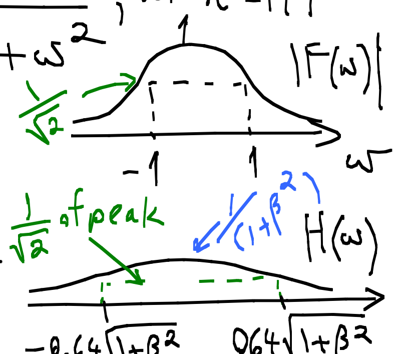
$$H(\omega) = \frac{\frac{1}{1+\omega^2}}{1+\frac{\beta^2}{1+\omega^2}} = \frac{\frac{1}{\cancel{1+\omega^2}}}{\frac{1+\beta^2+\omega^2}{\cancel{1+\omega^2}}} \Rightarrow H(\omega) = \frac{1}{(1+\beta^2)+\omega^2}$$

Peak value is $\frac{1}{1+\beta^2}$

According to the correspondence $e^{-\lambda|t|} \longleftrightarrow \frac{2\lambda}{\lambda^2+\omega^2}$, let $\lambda^2 = 1+\beta^2$

The impulse response is: $\boxed{h(t) = \frac{1}{2\sqrt{1+\beta^2}} e^{-\sqrt{1+\beta^2}\,|t|}}$

We can use feedback to increase the bandwidth, but with a compromise on the peak gain.



(c) Determine $v(t) = (f * g)(t)$ for the impulse response f of part (b).

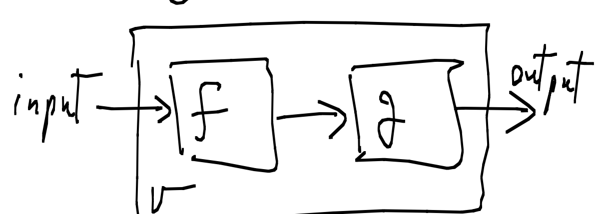
Fast Method: Think of F & G as being in cascade and do things in the frequency domain

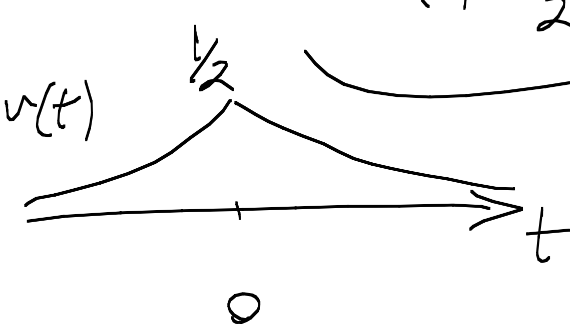
$$v(t) = (f * g)(t) \Rightarrow V(\omega) = F(\omega)G(\omega) = F(\omega)F^*(\omega) = |F(\omega)|^2$$

In part (b) we determined $|F(\omega)|^2 \Rightarrow V(\omega) = |F(\omega)|^2 = \frac{1}{1+\omega^2} \Rightarrow$

$$V(\omega) = \frac{1}{1+\omega^2} = \frac{1}{2}\,\frac{2}{1+\omega^2} \Rightarrow$$ Using the formula given, we have

$$v(t) = \frac{1}{2}e^{-|t|}$$





The function v is called the auto-correlation function of f. It peaks at $t=0$ and it is an even function.

MT2.3 (b): This was not required of you, but to find the 3 dB point for $H(\omega)$ we note that the peak value is at $\omega=0$, $H(0)=\frac{1}{1+\beta^2}$. Then solve for ω in the following equation $\frac{1}{(1+\beta^2)+\omega_0^2} \approx \frac{1}{\sqrt{2}}\frac{1}{1+\beta^2} \Rightarrow (1+\beta^2)+\omega_0^2=\sqrt{2}(1+\beta^2) \Rightarrow$
$\omega_0^2=(\sqrt{2}-1)(1+\beta^2) \Rightarrow \omega_0=\sqrt{\sqrt{2}-1}\,\sqrt{1+\beta^2} \approx 0.64\sqrt{1+\beta^2}$

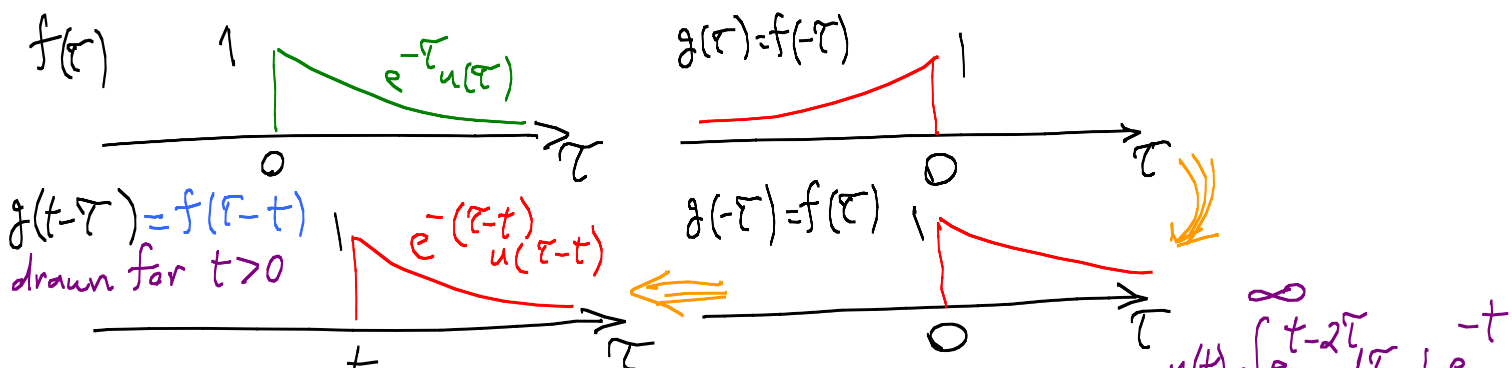
LAST Name ? FIRST Name ?

Lab Time ?

Problem	Points	Your Score
Name	10	5
1	45	45
2	15	15
3	45	45
Total	**115**	105

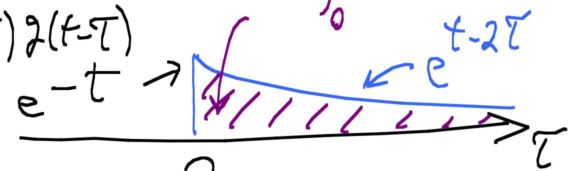
Sorry, Auntie: You forgot to write your name on this back page.

MT2.3(c): Slow method (brute force convolution... FUN!)

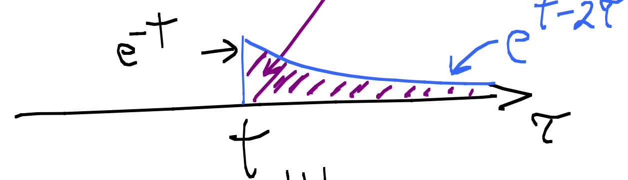


Case 1: $t<0$ $y(t)=\int_0^\infty e^{t-2\tau}\,d\tau=\frac{1}{2}e^{t}$

$f(\tau)g(t-\tau)$



Case 2: $t\geq 0$ $y(t)=\int_t^\infty e^{t-2\tau}\,d\tau=\frac{1}{2}e^{-t}$



Putting the two cases together: $y(t)=\frac{1}{2}e^{-|t|}$