MATH 54 FINAL

Dec 19 2014 8:00-11:00am

Your Name Student ID	SOLUTIONS

Section number and leader	

Do not turn this page until you are instructed to do so.

Show all your work in this exam booklet. There are pages with extra space at the end. No material other than simple writing utensils may be used. *In the event of an emergency or fire alarm leave your exam (closed) on your seat and meet with your GSI outside.*
If you need to use the restroom, leave your exam with your GSI while out of the room.

Your grade is determined from the following 5 problems, each of which has questions (a), (b), (c).

Each part of (a) yields either full or no credit, but you still have to show your work in calculations.

(b),(c) parts can yield partial credit, in particular for explanations and documentation of your approach, even when you don't complete the calculation. When asked to explain/show/prove, you should make clear and unambiguous statements, using a combination of formulas and words or arrows. (The graders will disregard formulas whose meaning is not stated explicitly.)

[7] **1(a)** Fill in the ... below.

$$\det\left(\begin{bmatrix}7&0&0\\8&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&2&3\\0&0&1\\4&5&6\end{bmatrix}\right) = \ldots \quad \underbrace{\det\begin{bmatrix}7&0&0\\8&1&0\\0&0&1\end{bmatrix}}_{\text{triangular}} \underbrace{\det\begin{bmatrix}1&2&3\\0&0&1\\4&5&6\end{bmatrix}}_{\text{expand by } 2^{nd} \text{ row}} = 7\cdot 3 = \underline{\underline{21}}$$

$$7 \qquad (-1)\det\begin{bmatrix}1&2\\4&5\end{bmatrix} = -(5-8) = 3$$

An eigenvalue of a square matrix A is ... a scalar λ so that $A\underline{x} = \lambda\underline{x}$ for some nonzero vector $\underline{x}$

The matrix product AB of an $m \times n$ matrix A and a $p \times q$ matrix B is defined under the condition ... $p = n$

as follows: ...

$$AB = A[\underline{b}_1 \ldots \underline{b}_q] = [A\underline{b}_1 \ldots A\underline{b}_q]$$

columns of B

Where $A\begin{bmatrix}b_1\\\vdots\\b_p\end{bmatrix} = [\underline{a}_1 \ldots \underline{a}_n]\begin{bmatrix}b_1\\\vdots\\b_p\end{bmatrix} = b_1\underline{a}_1 + \ldots + b_p\underline{a}_n$

columns of A

[6] **1(b)** Diagonalize the matrix $A = \begin{bmatrix} -7 & 0 & 6 \\ 0 & 5 & 0 \\ 6 & 0 & 2 \end{bmatrix}$ using the facts that $A \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = -10 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ and $\det(A - \lambda I) = (\lambda - 5)^2(\lambda + 10)$. (You should find P and D but need not compute P^{-1}.)

$$5\text{-eigenspace} = \mathrm{Nul}\begin{bmatrix} -7-5 & 0 & 6 \\ 0 & 5-5 & 0 \\ 6 & 0 & 2-5 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} -12 & 0 & 6 \\ 0 & 0 & 0 \\ 6 & 0 & -3 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 6 & 0 & -3 \end{bmatrix}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \; 6x_1 - 3x_3 = 0 \right\} = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

$$(-10)\text{-eigenspace} = \mathrm{span}\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \right\} \text{ is given}$$

$$\Rightarrow \quad \underline{\underline{P^{-1}AP = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -10 \end{bmatrix} = D}} \quad \text{for } \underline{\underline{P = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix}}}$$

[4] **1(c)** Give the definition for the claim that the vectors $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ are linearly independent.

(Hint: The definition uses linear combinations.)
Then translate this claim into a statement about a matrix and verify it. State any theorems that you use.

Def^{n}: The vectors are linearly independent if

$c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \underline{0}$ holds only for $c_1 = c_2 = c_3 = 0$.

$\Updownarrow$

$\begin{bmatrix} -1 & 1 & 3 \\ 1 & 2 & 4 \\ 0 & 3 & 5 \end{bmatrix} \underline{c} = \underline{0}$ only has solution $\underline{c} = \underline{0}$

$\Updownarrow$ row equivalent matrices have the same solutions (of homogeneous equation)

$\begin{bmatrix} -1 & 1 & 3 \\ 0 & 3 & 7 \\ 0 & 0 & -2 \end{bmatrix}$ has no nontrivial solutions because every column has a pivot, so there are no free variables.

space for extra work - label by problem number and write “XTRA” on the page of actual problem

[8] **2(a)** Fill in the ... below.

The kernel of a linear transformation $T : V \to W$ is ... the set of vectors v in V with T(v) = 0 – the zero vector in W.

The dimension of a vector space V is ... the number of vectors in a basis of V, and ∞ if V has no finite spanning set.

A vector space V is a nonempty set of vectors, on which addition and multiplication by scalars is defined (and closed), which contains a zero vector $\mathbf{0}$ so that ... u+0 = u for each u

and on which the following axioms hold for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and scalars c, d.

$$\mathbf{u}+\mathbf{v} = v+u \qquad (\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w}) \qquad c(\mathbf{u}+\mathbf{v}) = cu+cv$$

$$(c+d)\mathbf{u} = c\mathbf{u}+d\mathbf{u} \qquad c(d\mathbf{u}) = (cd)\mathbf{u} \qquad 1\mathbf{u} = \mathbf{u}$$

[6] **2(b)** Give the definition for the change-of-coordinate matrix $P = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ from a basis $\mathcal{B}$ to a basis $\mathcal{C}$ of a vector space V in terms of the coordinate maps $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ and $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{C}}$.

For $V = \mathbb{P}_2$, find the change-of-coordinate matrix P from $\mathcal{B} = \{\underset{b_1}{1}, \underset{b_2}{t}, \underset{b_3}{t^2}\}$ to $\mathcal{C} = \{\underset{c_1}{1+t^2}, \underset{c_2}{1-t^2}, \underset{c_3}{2t}\}$.

Hint: You can check your results at $[2+3t+4t^2]_{\mathcal{C}} = P\begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}\cancel{3}\ 3\\-1\\3/2\end{bmatrix}$.

$\underline{\text{Def}^{\underline{n}}}$: $P[x]_{\mathcal{B}} = [x]_{\mathcal{C}}$

$P\begin{bmatrix}1\\0\\0\end{bmatrix} = [b_1]_{\mathcal{C}} = [1]_{\mathcal{C}} = \begin{bmatrix}1/2\\1/2\\0\end{bmatrix}$ since $1 = \frac{1}{2}(\underbrace{1+t^2}_{c_1}) + \frac{1}{2}(\underbrace{1-t^2}_{c_2})$

$P\begin{bmatrix}0\\1\\0\end{bmatrix} = [b_2]_{\mathcal{C}} = [t]_{\mathcal{C}} = \begin{bmatrix}0\\0\\1/2\end{bmatrix}$ since $t = \frac{1}{2}\cdot\underbrace{2t}_{c_3}$

$P\begin{bmatrix}0\\0\\1\end{bmatrix} = [b_3]_{\mathcal{C}} = [t^2]_{\mathcal{C}} = \begin{bmatrix}1/2\\-1/2\\0\end{bmatrix}$ since $t^2 = \frac{1}{2}(\underbrace{1+t^2}_{c_1}) - \frac{1}{2}(\underbrace{1-t^2}_{c_2})$

$$\Rightarrow \underline{\underline{P = \begin{bmatrix}1/2 & 0 & 1/2\\1/2 & 0 & -1/2\\0 & 1/2 & 0\end{bmatrix}}}$$

CHECK: $[2+3t+4t^2]_{\mathcal{C}} = P[\underbrace{2+3t+4t^2}_{2b_1+3b_2+4b_3}]_{\mathcal{B}} = \begin{bmatrix}1/2 & 0 & 1/2\\1/2 & 0 & -1/2\\0 & 1/2 & 0\end{bmatrix}\begin{bmatrix}2\\3\\4\end{bmatrix}$

$= \begin{bmatrix}2/2+4/2\\2/2-4/2\\3/2\end{bmatrix} = \begin{bmatrix}3\\-1\\3/2\end{bmatrix}$

$\underline{\text{indeed}}$: $3(1+t^2) - 1(1-t^2) + \frac{3}{2}\cdot 2t$

$= 3 + 3t^2 - 1 + t^2 + 3t$

$= 2 + 3t + 4t^2$

$\underline{\text{alternative}}$:

$P^{-1} = \left[\begin{array}{ccc|ccc}1 & 1 & 0 & 1 & 0 & 0\\0 & 0 & 2 & 0 & 1 & 0\\1 & -1 & 0 & 0 & 0 & 1\end{array}\right]$ $\quad \begin{array}{cccccc}1 & 1 & 0 & 1 & 0 & 0\\0 & 0 & 1 & 0 & 1/2 & 0\\0 & -2 & 0 & -1 & 0 & 1\end{array}$ $\quad \begin{array}{ccc}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{array}\begin{bmatrix}1/2 & 0 & 1/2\\1/2 & 0 & -1/2\\0 & 1/2 & 0\end{bmatrix} = P$

[6] **2(c)** Let H be a subspace of a finite dimensional vector space V such that there is a vector $\mathbf{w}$ in V that does not lie in H. State and explain the implications about the dimensions of H and V.

$$\boxed{\dim H < \dim V}$$

Let $\{v_1 \dots v_p\}$ be a basis of H (exists since subspaces of finite dimensional vector spaces are finite dimensional)

Then $\{v_1 \dots v_p, w\}$ is linearly independent in V.

$$c_0 w + c_1 v_1 + \dots + c_p v_p = 0$$

$$\Rightarrow c_0 = 0 \quad \text{or} \quad \underbrace{w = -\frac{c_1}{c_0} v_1 \dots - \frac{c_p}{c_0} v_p \text{ in } H}_{\text{but } w \text{ not in } H}$$

$$\Downarrow$$

$$c_0 = 0, \; c_1 v_1 + \dots + c_p v_p = 0 \xRightarrow[\text{lin. indep.}]{v_1 \dots v_p} c_0 = c_1 = \dots = c_p = 0$$

Since V is finite dimensional, this is only possible for $p+1 \leq \dim V$.

But $p = \dim H$, so $\dim H < p+1 \leq \dim V$.

space for extra work - label by problem number and write “XTRA” on the page of actual problem

[8] **3(a)** Fill in the ... below.

The general solution of $y'' - 10y' + 25y = 0$ is $y(t) = ..c_1 e^{5t} + c_2 t e^{5t}$

$r^2 - 10r + 25 = 0$

$\Leftrightarrow r = 5 \pm \sqrt{5^2 - 25} = 5$ double root

$y'' - 10y' + 25y = e^{5t}$ has a particular solution of the form (with undetermined coefficients)
$y(t) = ...c\, t^2 e^{5t}$

$3y'' - 25y = 7e^{-t}\sin(2t)$ has a particular solution of the form (with undetermined coefficients)
$y(t) = ... A e^{-t} \cos 2t + B e^{-t} \sin 2t$

or $Re(C e^{(-1+2i)t})$

A third order ODE of the form $\frac{d^3y}{dt^3} + ay'' + by' + cy = 0$ has a unique solution if we specify the initial values $y(0)$, ... $y'(0), y''(0)$

[6] **3(b)** Find the solution to $y'' - 25y = 16\cos(3t)$ with initial values $y(0) = 1, y'(0) = 10$, using a (real or complex) method of undetermined coefficients.

$r^2 - 25$ has roots ± 5

$\to$ homogeneous sol$^{\underline{ns}}$ e^{5t}, e^{-5t}

particular Ansatz $y_p(t) = A\cos 3t + B\sin 3t$

$y_p'' = -9A\cos 3t - 9B\sin 3t$

plug in: $y_p'' - 25y_p = (-9-25)A\cos 3t + (-9-25)B\sin 3t \overset{!}{=} 16\cos 3t$

$\Rightarrow -34A = 16 \quad , -34B = 0$

$\Rightarrow A = -\frac{8}{17} \quad , B = 0$

$\Rightarrow y_p(t) = -\frac{8}{17}\cos 3t$

$\Rightarrow$ general sol$^{\underline{n}}$ $y(t) = \frac{-8}{17}\cos 3t + c_1 e^{5t} + c_2 e^{-5t}$

$y(0) = \frac{-8}{17} + c_1 + c_2 = 1 \quad \Rightarrow c_1 + c_2 = \frac{25}{17}$

$y'(0) = 5c_1 - 5c_2 = 10 \quad \Rightarrow c_1 - c_2 = 2$

$\Rightarrow 2c_1 = 2 + \frac{25}{17} = \frac{59}{17} \quad , \quad 2c_2 = \frac{25}{17} - 2 = \frac{-9}{17}$

$$\Rightarrow \underline{\underline{y(t) = -\frac{8}{17}\cos 3t + \frac{59}{34}e^{5t} - \frac{9}{34}e^{-5t}}}$$

[6] **3(c)** Set up the variation of parameters formulas which solve $y'' - 25y = 16\cos(3t)$ by going through the following steps:

- The variation of parameters Ansatz is $y(t) = c_1(t)$ ....e^{5t}.... $+ c_2(t)$ ....e^{-5t}....

- We impose a convenient extra requirement for the functions c_1, c_2:

$$c_1' e^{5t} + c_2' e^{-5t} = 0$$

- Plugging this Ansatz and condition into the ODE yields an equation for c_1' and c_2':

$$c_1' y_1' + c_2' y_2' = 16 \cos 3t$$

$$y_1(t) = e^{5t} \Rightarrow y_1' = 5e^{5t}$$
$$y_2(t) = e^{-5t} \Rightarrow y_2' = -5e^{-5t}$$

- We can rewrite the ODE and extra requirement in matrix form

$$\underbrace{\begin{bmatrix} e^{5t} & e^{-5t} \\ 5e^{5t} & -5e^{-5t} \end{bmatrix}}_{\det = -5-5=-10} \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 16\cos 3t \end{bmatrix}$$

$$\to \left[\;\right]^{-1} = \frac{-1}{10}\begin{bmatrix} -5e^{-5t} & -e^{-5t} \\ -5e^{5t} & e^{5t} \end{bmatrix}$$

- Solving this linear system for c_1', c_2' leads to the integral formulas
(these need not be solved but should be so explicit that they can be plugged into a symbolic calculator)

$$c_1(t) = c_1(0) + \int_0^t \dots \frac{-1}{10}\left(-e^{-5t} \cdot 16\cos 3t\right)$$

$$c_2(t) = c_2(0) + \int_0^t \dots \frac{-1}{10}\left(e^{5t} \cdot 16\cos 3t\right)$$

space for extra work - label by problem number and write “XTRA” on the page of actual problem

[7] **4(a)** Fill in the ... below.

If $\mathbf{A}\begin{bmatrix}1\\2\end{bmatrix} = 5\begin{bmatrix}1\\2\end{bmatrix}$ and $\mathbf{A}\begin{bmatrix}3\\1\end{bmatrix} = -7\begin{bmatrix}3\\1\end{bmatrix}$, then the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$\mathbf{x}(t) = \ldots$ $c_1 e^{5t}\begin{bmatrix}1\\2\end{bmatrix} + c_2 e^{-7t}\begin{bmatrix}3\\1\end{bmatrix}$

If the vector functions $\mathbf{y}_1, \mathbf{y}_2$ solve $\mathbf{y}_1' = \mathbf{A}\mathbf{y}_1 + \begin{bmatrix}0\\3\sin t\end{bmatrix}$ and $\mathbf{y}_2' = \mathbf{A}\mathbf{y}_2 + \begin{bmatrix}2e^{2t}\\e^{2t}\end{bmatrix}$,

then $\mathbf{x} = \mathbf{y}_1 - \mathbf{y}_2$ solves the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \begin{bmatrix} -2e^{2t} \\ 3\sin t - e^{2t} \\ \ldots \end{bmatrix}.$$

$A = \begin{bmatrix}-7 & 0\\0 & 5\end{bmatrix}$ has matrix exponential function $e^{tA} = \ldots$ $\begin{bmatrix} e^{-7t} & 0 \\ 0 & e^{5t}\end{bmatrix}$

[7] **4(b)** Find the general solution of $\mathbf{x}'(t) = \overbrace{\begin{bmatrix} -7 & 0 \\ 0 & 5 \end{bmatrix}}^{=A} \mathbf{x}(t) + \begin{bmatrix} e^{-5t} \\ e^{5t} \end{bmatrix}$ using variation of parameters.

Ansatz: $\underline{x}(t) = e^{tA}\, \underline{c}(t)$

plug in: $\underline{x}' = A e^{tA} \underline{c}(t) + e^{tA} \underline{c}'(t) \overset{!}{=} A e^{tA} \underline{c}(t) + \begin{bmatrix} e^{-5t} \\ e^{5t} \end{bmatrix}$

$$\Rightarrow \quad \underline{c}'(t) = e^{-tA} \begin{bmatrix} e^{-5t} \\ e^{5t} \end{bmatrix} = \begin{bmatrix} e^{7t} & 0 \\ 0 & e^{-5t} \end{bmatrix} \begin{bmatrix} e^{-5t} \\ e^{5t} \end{bmatrix} = \begin{bmatrix} e^{2t} \\ 1 \end{bmatrix}$$

solve: $\underline{c}(t) = \begin{bmatrix} \frac{1}{2} e^{2t} + c_1 \\ t + c_2 \end{bmatrix}$

plug in: $\underline{x}(t) = e^{tA}\, \underline{c}(t) = \begin{bmatrix} e^{-7t} & 0 \\ 0 & e^{5t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} e^{2t} + c_1 \\ t + c_2 \end{bmatrix}$

$$= \underline{\underline{\begin{bmatrix} \frac{1}{2} e^{-5t} + c_1 e^{-7t} \\ t e^{5t} + c_2 e^{5t} \end{bmatrix}}}$$

(Alternative (max half credit): System splits so simpler ODE methods yield

$x' = -7x + e^{-5t} \rightsquigarrow x(t) = \frac{1}{2} e^{-5t} + c_1 e^{-7t}$

$y' = 5y + e^{5t} \rightsquigarrow y(t) = t e^{5t} + c_2 e^{5t}$)

[6] **4(c)** Let $\mathbf{A} : \mathbb{R} \to M_{n\times n}$ be a continuous matrix function.

Give the definition of a fundamental matrix $\mathbf{X}(t)$ for the system $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t)$.

Give and prove a general formula for solving initial value problems $\quad \mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t), \quad \mathbf{x}(t_0) = \mathbf{v}$ in terms of $\mathbf{X}(t)$, t_0, and $\mathbf{v}$.

Defn: $\quad \mathbf{X}'(t) = A(t)\mathbf{X}(t) \quad , \; \mathbf{X}(t_0)$ invertible for some (and hence all) t_0

Claim: $\quad \underline{x}(t) = \mathbf{X}(t)\,\mathbf{X}(t_0)^{-1}\underline{v} \quad$ solves

Proof: $\quad \underline{x}'(t) = \mathbf{X}'(t)\,\mathbf{X}(t_0)^{-1}\underline{v} = A(t)\underbrace{\mathbf{X}(t)\,\mathbf{X}(t_0)^{-1}\underline{v}}_{\underline{x}(t)}$

$$\underline{x}(t_0) = \underbrace{\mathbf{X}(t_0)\,\mathbf{X}(t_0)^{-1}}_{I}\underline{v} = \underline{v}$$

space for extra work - label by problem number and write “XTRA” on the page of actual problem

[7] **5(a)** Fill in the ... below.

The Fourier series of a smooth function f on $[-5,5]$ is $\frac{1}{2}a_0+\sum_{n=1}^{\infty} a_n \cos(\ldots \frac{\pi}{5}n \ldots\, x)+b_n \sin(\ldots \frac{\pi}{5}n \ldots\, x)$

with

$$a_n = \ldots \frac{\int_{-5}^{5} f(x)\cos\frac{\pi}{5}nx}{\int_{-5}^{5} \cos^2\frac{\pi}{5}nx} = \frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{\pi}{5}nx$$

$$b_n = \ldots \frac{\int_{-5}^{5} f(x)\sin\frac{\pi}{5}nx}{\sin^2\frac{\pi}{5}nx} = \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{\pi}{5}nx$$

The Fourier coefficients of $f(x) = \overbrace{2\sin^2(x) + \underbrace{\sqrt{2}}_{b_5}\sin(\underbrace{\pi x}_{\frac{\pi}{5}\cdot 5x}) + 2\cos^2(x)}^{=2\ =\frac{1}{2}a_0} + \underbrace{7}_{a_3}\cos\left(\frac{3\pi}{5}x\right)$ on $[-5,5]$ are

$a_0 = \ldots 4$ $\quad a_1 = \ldots 0$ $\quad a_2 = \ldots 0$ $\quad a_3 = \ldots 7$ $\quad a_4 = \ldots 0$ $\quad a_5 = \ldots 0$

$b_1 = \ldots 0$ $\quad b_2 = \ldots 0$ $\quad b_3 = \ldots 0$ $\quad b_4 = \ldots 0$ $\quad b_5 = \ldots \sqrt{2}$ $\quad b_6 = \ldots 0$

[7] **5(b)** Find the solution to the initial-boundary value problem (by any method)

$$\frac{\partial u}{\partial t} = \sqrt{2}\,\frac{\partial^2 u}{\partial x^2} \qquad \text{for} \quad 0 < x < \pi,\ t > 0, \qquad \text{(PDE)}$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0 \qquad \text{for} \quad t > 0, \qquad \text{(bc)}$$

$$u(x,0) = \sum_{n=0}^{\infty} 3^{-n}\cos(nx) \qquad \text{for} \quad 0 < x < \pi. \qquad \text{(ic)}$$

Fourier Ansatz $u(x,t) = \sum_{n=0}^{\infty} T_n(t)\cos(nx)$ solves (bc)

(PDE): $\sum_{n=0}^{\infty} T_n' \cos nx = -\sqrt{2}\sum_{n=0}^{\infty} T_n n^2 \cos nx$

$\iff T_n' = -\sqrt{2}\, n^2 T_n$

(ic): $\sum_{n=0}^{\infty} T_n(0)\cos nx = \sum_{n=0}^{\infty} 3^{-n}\cos nx$

$\iff T_n(0) = 3^{-n}$

$\Rightarrow T_n(t) = 3^{-n} e^{-\sqrt{2}n^2 t}$

$$\Rightarrow \quad \underline{\underline{u(x,t) = \sum_{n=0}^{\infty} 3^{-n} e^{-\sqrt{2}n^2 t}\cos(nx)}}$$

<u>alternative</u>: General solution

$u(x,t) = \sum_{n=0}^{\infty} c_n e^{-\sqrt{2}n^2 t}\cos(nx)$ from separation of variables

(ic): $\sum_{n=0}^{\infty} c_n \cos nx = \sum_{n=0}^{\infty} 3^{-n}\cos nx$

$\Rightarrow c_n = 3^{-n} \Rightarrow u(x,t) = \sum_{n=0}^{\infty} 3^{-n} e^{-\sqrt{2}n^2 t}\cos(nx)$

[6] **5(c)** Determine ODE's and initial conditions (but don't solve them) for the coefficients $B_0(t), B_1(t), B_2(t), \ldots$ of any function $u(x,t) = B_0(t) + \sum_{n=1}^{\infty} B_n(t)\sin(nx)$ that solves

$$\text{(PDE)} \qquad \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + e^{-t} + 4t\sin(2x) \qquad \text{for} \quad 0 < x < 2\pi,\ t > 0,$$

$$\text{(i)} \quad u(x,0) = 3 + \sin(x), \qquad \text{(ii)} \ \frac{\partial u}{\partial t}(x,0) = 5\sin(2x) \qquad \text{for} \quad 0 < x < 2\pi$$

(PDE): $B_0'' + \sum_{n=1}^{\infty} B_n'' \sin nx = -\sum_{n=1}^{\infty} B_n n^2 \sin nx + e^{-t} + 4t \sin 2x$

$\iff B_0'' = e^{-t},\ B_2'' = -4B_2 + 4t,\ B_n'' = -n^2 B_n$ for $n = 1, 3, \ldots$

(i): $u(x,0) = B_0(0) + \sum_{n=1}^{\infty} B_n(0) \sin nx = 3 + \sin x$

$\iff B_0(0) = 3,\ B_1(0) = 1,\ B_n(0) = 0$ for $n = 2, 3, \ldots$

(ii): $\frac{\partial u}{\partial t}(x,0) = B_0'(0) + \sum_{n=1}^{\infty} B_n'(0) \sin nx = 5 \sin 2x$

$\iff B_2'(0) = 5,\ B_n'(0) = 0$ for $n \neq 2$

$B_0'' = e^{-t}$	$B_1'' = -B_1$	$B_2'' = -4B_2 + 4t$	$B_n'' = -n^2 B_n$
$B_0(0) = 3$	$B_1(0) = 1$	$B_2(0) = 0$	$B_n(0) = 0$
$B_0'(0) = 0$	$B_1'(0) = 0$	$B_2'(0) = 5$	$B_n'(0) = 0$

space for extra work - label by problem number and write “XTRA” on the page of actual problem

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