NAME: 3 May 2004

Physics 112

Spring 2004

Midterm 3 Solution.

(50 minutes =50 points)

1. Thermodynamic Identity (5 points)

Write down the thermodynamic identity for the enthalpy

$$H = U + PV$$

What are the natural variables for H?

$$dH = dU + \cancel{dp} + Vdp \quad \leftarrow dU = \tau d\sigma - pdV + \mu dN$$

$$= \tau d\sigma + Vdp + \mu dN$$

$$\therefore \quad H = H(\sigma, p, N)$$

2. Light bulb problem (10 points)

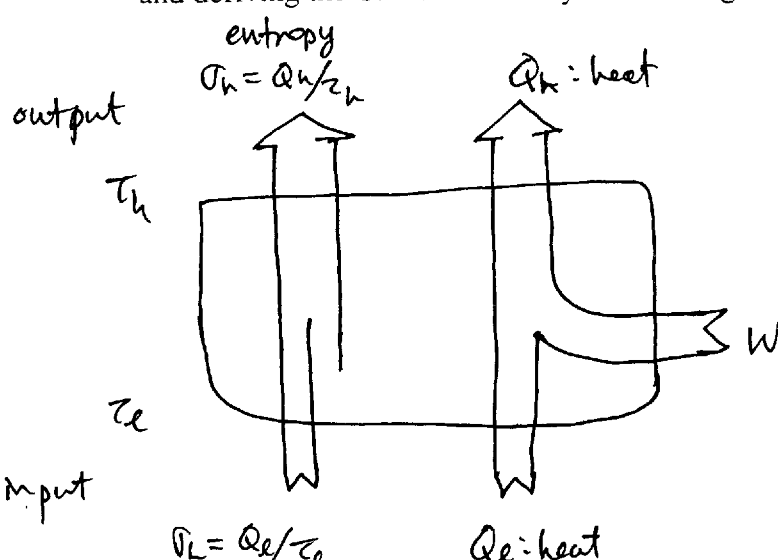
A 100W light bulb is left burning inside a reversible refrigerator that draws 100W.

a) (5 points) Can the refrigerator cool below room temperature?

We have a situation $Q_\ell = W$.

$$\therefore \quad W = \frac{\tau_h - \tau_\ell}{\tau_\ell} \cdot Q_\ell \Rightarrow \tau_\ell = \frac{1}{2}\tau_h \quad \therefore \text{Yes.}$$

b) (5 points) Justify your answer by drawing the exchanges of energy and entropy and deriving the Carnot efficiency of the refrigerator.



There is no entropy generation inside.

$$\therefore \sigma_h = \sigma_\ell \Rightarrow Q_h = Q_\ell \cdot \frac{\tau_h}{\tau_\ell}.$$

and $W = Q_h - Q_\ell = \left(\frac{\tau_h - \tau_\ell}{\tau_\ell}\right) Q_\ell$.

$\therefore$ carnot efficiency of refrigerator

$$\gamma_c = \frac{Q_\ell}{W} = \frac{\tau_\ell}{\tau_h - \tau_\ell} = 1.$$

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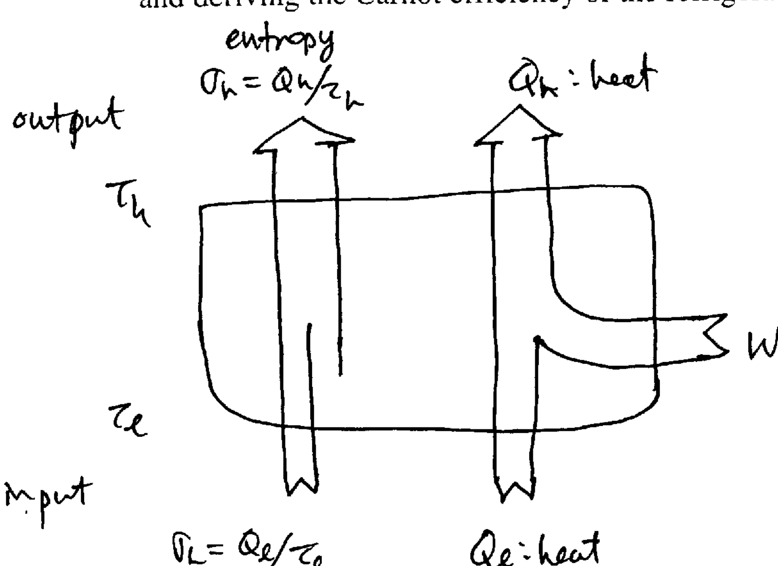

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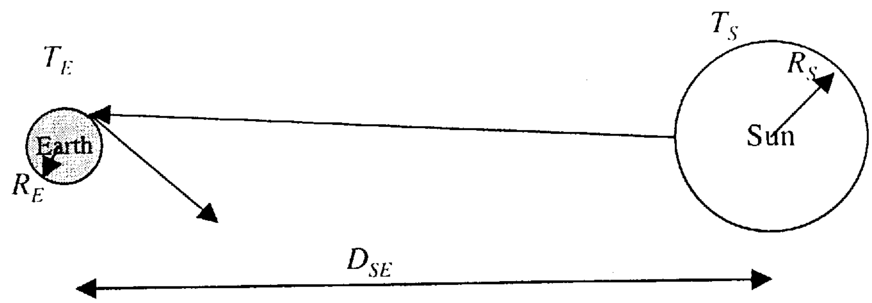
3. **Greenhouse effect** (15 points)

We consider the radiation balance between the earth and the sun, assumed to be both perfect black body radiators.

a)(5 points) We assume first that there are no greenhouse gases in the atmosphere. Knowing the radius of the sun (R_S=7x10^{8}m), the mean earth-sun distance (D_{SE}= 1.50x10^{11}m) and the temperature of the sun (T_S=5800K) compute the mean temperature T_E of the earth. The radius R_E of the earth drops out of the final formula.



- Total power radiated by sun = $(\sigma_B T_S^4)\cdot 4\pi R_S^2$.
- Total power received by earth with cross section πR_E^2

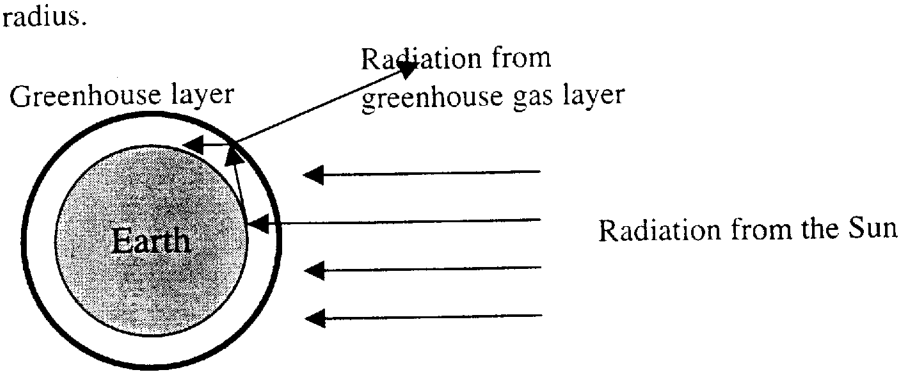
$$= \sigma_B T_S^4 \cdot 4\pi R_S^2 \cdot \frac{\pi R_E^2}{4\pi D_{SE}^2} \quad -①$$

- Total power radiated by Earth $= (\sigma_B T_E^4)\cdot 4\pi R_E^2 \quad -②$

① = ② ∴ $T_E = T_S\sqrt{\frac{R_S}{2D_{SE}}}$

$= 280\,K$

b) (10 points) We introduce now a greenhouse gas layer, very close to the surface of the earth. We will assume that the greenhouse layer does not absorb the (mainly visible) solar radiation but fully absorbs (and reemits over the 4π solid angle) the (infrared) radiation reemitted by the earth. By writing down the energy flux balance for the greenhouse layer and the earth separately, compute the temperature now reached by the earth. We assume that the black body formulae apply. The greenhouse layer is very close to the earth, when compared to the earth radius.



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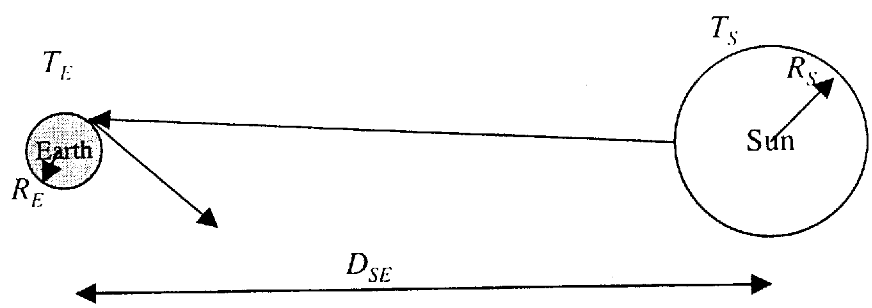

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- Total power radiated by Earth $= (\sigma_B T_E^4) \cdot 4\pi R_E^2 \quad \sim ②$

① = ②

$$\therefore \; T_E = T_s \sqrt{\frac{R_s}{2D_{SE}}} = 280\,K$$

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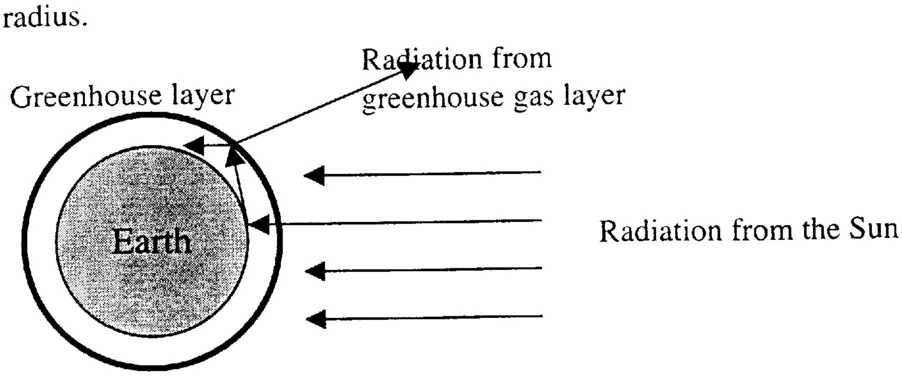
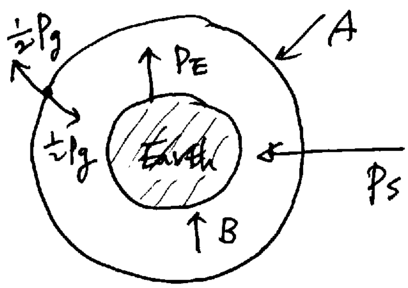




Now we have three different power sources, P_s (from the Sun), P_E (from the Earth), and P_g (from the greehouse layer).

At layer A, we have $P_s = \frac{1}{2}P_g$

" B, " $P_s = P_E - \frac{1}{2}P_g$

4. **Pressure of a Fermi-Dirac gas (20 points)** $\quad \therefore P_E = 2P_s \Rightarrow T_E = 2^{\frac{1}{4}} \cdot T_s \sqrt{\frac{R_s}{2D_{SE}}} = 333\,K$

The purpose of this problem is to evaluate from first principles the pressure of a Fermi-Dirac gas (i.e., *not* using $P = -\left.\frac{\partial F}{\partial V}\right|_{\tau, N_i}$). We assume that $\tau << \varepsilon_F$ (the Fermi energy i.e, the chemical potential at zero temperature).

a. (2 points) How is ε_F determined in function of the particle density n and the density of state $D(\varepsilon)$?

$$N = \int_0^\infty \frac{1}{e^{(\epsilon-\mu)/\tau}+1} \Delta(\varepsilon)\,d\varepsilon \cdot \underbrace{d^3x}_{V}$$

$$\therefore n = \frac{N}{V} = \int_0^\epsilon \frac{1}{e^{(\epsilon-\mu)/\tau}+1} \Delta(\varepsilon)\,d\varepsilon \xrightarrow[\tau=0]{} \int_0^{\epsilon_F} \Delta(\varepsilon)\,d\varepsilon$$

b. (5 points) From what you know about the density of states in phase space, show that the density of states $D(\varepsilon)$ of a non relativistic Fermi-Dirac gas can be expressed as

$$D(\varepsilon)dVd\varepsilon = 3/2\frac{n\varepsilon^{1/2}}{\varepsilon_F^{3/2}}dVd\varepsilon$$

dV is the volume element (this is the convention of the notes which is more customary than that of the book where the volume V is put inside $D(\varepsilon)$. (Hint: To derive rapidly this expression, you only need to know that $D(\varepsilon)$ is proportional to $\sqrt{\varepsilon}$.)

From the hint, we have $\mathcal{D}(\varepsilon)\,d\varepsilon \sim p^2 dp \sim \sqrt{\epsilon}\,d\epsilon$.

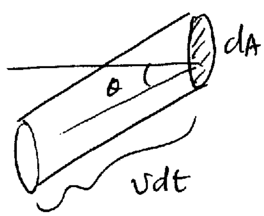
$$\therefore n = \int_0^{\epsilon_F} C \cdot \sqrt{\epsilon}\,d\epsilon = C \cdot \frac{2}{3}\epsilon_F^{3/2} \qquad C: \text{constant.}$$

$$\therefore C = \frac{3}{2} \cdot \frac{n}{\epsilon_F^{3/2}}$$

$$\Rightarrow \mathcal{D}(\varepsilon)\,dV\,d\varepsilon = \frac{3}{2} \cdot \frac{n \cdot \epsilon^{1/2}}{\epsilon_F^{3/2}}\,dV\,d\epsilon$$

c. (5 points) Show that the flux of particles of energy $\varepsilon<\varepsilon_F$ incident at angle θ within a solid angle $d\Omega$ on a wall surface area dA is

$$F(\varepsilon,\cos\theta)dAd\Omega d\varepsilon = \frac{3}{8\pi}\frac{n\varepsilon^{1/2}}{\varepsilon_F^{3/2}} \cdot v\cos\theta d\Omega \, dA \, d\varepsilon$$



the volume element is $dV = v dt \cdot dA \cdot \cos\theta$

To get the flux of particles within a solid angle $d\Omega$ on a wall surface dA, we need multiply $\frac{d\Omega}{4\pi}$ and divide dt

$$\therefore F(\epsilon, \cos\theta) dA d\Omega d\epsilon = \frac{3}{8\pi} \cdot \frac{n\epsilon^{1/2}}{\epsilon_F^{3/2}} \cdot v\cos\theta \cdot d\Omega \cdot d\epsilon$$

d. (5 points) Show that the pressure is

$$P = \frac{2}{5} n\varepsilon_F$$

You may want to use the fact that the force exerted by the gas particles on a wall is equal to the sum of the total momentum change per unit time of the particles incident on the surface. The scattering is assumed to be specular (symmetric with respect to the surface normal).

Momentum transfer of a particle is a force $F = 2p\cos\theta$.

So we have the pressure $\text{Pressure} = \frac{\langle \delta F \rangle}{\delta A}$

$$\therefore \text{Pressure} = \frac{\langle \delta p \rangle}{\delta t} \cdot \frac{1}{\delta A} = \frac{\langle \delta F \rangle}{\delta A} = \int d\epsilon d\Omega \cdot 2p\cos\theta \cdot F(\epsilon, \cos\theta)$$

$$= \int_0^{\epsilon_F} d\epsilon \cdot \frac{3}{2} \cdot \frac{n\epsilon^{1/2}}{\epsilon_F^{3/2}} \cdot 2pv \int d\Omega \frac{\cos^2\theta}{4\pi} = \frac{2}{5} n\epsilon_F$$

(we used $2pv = 4\epsilon$, $\int d\Omega \cos^2\theta \frac{1}{4\pi} = \frac{1}{6}$)

e. (3 points) Show that as for all non-relativistic gases, the pressure is 2/3 of the energy density.

$$= \int \epsilon D(\epsilon) d\epsilon$$

energy density $u = n \int \epsilon \cdot \frac{3}{2} \cdot \frac{\epsilon^{1/2}}{\epsilon_F^{3/2}} d\epsilon = \frac{3}{5} n\epsilon_F$

$$\therefore P = \text{Pressure} = \frac{2}{3} u.$$