1) 9
2) 6
3) 10
4) 10
5) 10
6) 10
Total) 55

Name: ______ SID: ______ GSI: ______

Math 53 Midterm #1, 2/26/14, 3:40 PM – 5:00 AM
(no exams will be accepted between 4:50 and 5:00)

No notes or electronic devices are permitted. Please write your name, SID number, and GSI's name on each sheet; **pages without identifying information will not be graded.** All answers must be justified to receive credit. There are six pages, and each page is worth 10 points. Please put a box around your final answer and cross out incorrect work. Good luck!

1. (10 points) Find the point on the curve $\mathbf{r}(t) = \langle 1, t, t^2 \rangle$ where the tangent line is parallel to the plane $4x + 5y + 6z = 0$.

$\vec{r}'(t) = \langle 0, 1, 2t \rangle$

Normal vector of plane: $\langle 4, 5, 6 \rangle$

$\langle 4, 5, 6 \rangle \cdot \langle 0, 1, 2t \rangle = 0$ ← justify (-1)

$5 + 12t = 0$

$t = -\frac{5}{12}$

$\boxed{\text{Point: } \left(1, -\frac{5}{121}, \frac{25}{144}\right)}$

~~$\langle 4, 5, 6 \rangle$~~

~~$\langle 0, 1, 2t \rangle$~~

~~$5 + 12t = 0$~~

~~$t = -5/12$~~

Name: __________ SID: __________ GSI: __________

2. (Short questions, 5 points each)

   (a) Write the equation of a quadric surface containing the parametrized curve
$$(x(t), y(t), z(t)) = (t^3, t^4, t^5).$$
(Your answer should be a quadratic equation in x, y, and z.) What kind of quadric surface is it? (e.g. ellipsoid, hyperboloid of one sheet, etc.)

   (b) Suppose that f is a function of x and y such that $f_x = x + 2y$ and $f_y = ax + 3y$ where a is a constant. What does a have to be, and why?

a) ~~$x = t^3$, $y = t^4$, $z = t^5$~~

~~$\boxed{z^2 = x^2 y}$~~ ~~This is a cone~~

~~$t^{10} = t^6 t^4$~~ ✓

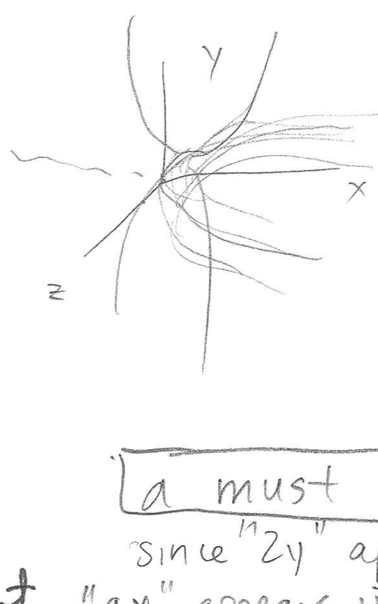
✓

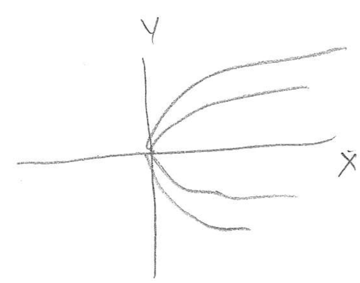
$\boxed{y^2 = xz}$ This is a ~~paraboloid~~. ✗

correct answer cone

$t^8 = t^3 t^5 = t^8$ ✓

2





b) $\frac{\partial f}{\partial x} = x + 2y$ $\qquad$ $\frac{\partial f}{\partial y} = ax + 3y$

$\frac{1}{2}x^2 + 2yx$ $\qquad$ $axy + \frac{3}{2}y^2$

$f(x,y) = \frac{1}{2}x^2 + 2yx + \frac{3}{2}y^2 + C$ ← not constant

$f_y = 2x + 3y = ax + 3y \rightarrow a = 2$

$\boxed{\text{a must be 2}}$ because since "2y" appears in f_x and "ax" appears in f_y, the actual function must include axy, where a is the constant. If we let $g(x,y) = axy$, $\frac{\partial g}{\partial x} = ay = 2y$, so $\boxed{a = 2}$ ✓

4

+10

Name: ______ SID: ______ GSI: ______

3. (10 points) Let P be the tangent plane to the surface

$$x^2 + y^2 + xyz = 11$$

at the point $(1, 2, 3)$. The plane P intersects the x-axis at a point $(a, 0, 0)$. What is a?

$F = x^2 + y^2 + xyz$ $\qquad$ $ax + by + cz = d$

$F_x = 2x + yz$, $F_x(1,2,3) = 2 + 6 = 8$ $\quad$ +3

$F_y = 2y + xz$, $F_y(1,2,3) = 4 + 3 = 7$ $\quad$ +1

$F_z = xy$, $F_z(1,2,3) = 2$

$8x + 7y + 2z = d$

+3 $8(1) + 7(2) + 2(3) = 28$ $\quad$ tangent plane.

$8 + 14 + 6 = 28$

$8x + 7y + 2z - 28 = 0$

Plug in $(a, 0, 0)$

$8a = 28$

$a = \frac{28}{8} = \frac{7}{2}$ $\quad$ +3

Name: ______________ SID: ______________ GSI: ______________

4. (10 points) Does the limit exist, and if so what is the limit?

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} =?$$

Justify your answer.

Approach $y \to 0$: $\lim_{x\to 0}\left(\lim_{(x,y)\to(x,0)} \frac{x^3+y^3}{x^2+y^2}\right) = \lim_{x\to 0} \frac{x^3}{x^2} = \lim_{x\to 0} x = 0.$

along y-axis

Use polar coordinates: $x = r\cos\theta$

$y = r\sin\theta$

$$\lim_{r\to 0} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \lim_{r\to 0} \frac{r^3(\cos^3\theta + \sin^3\theta)}{r^2(\cos^2\theta + \sin^2\theta)}$$

$$= \lim_{r\to 0} \underbrace{r}_{\text{approaches } 0} \underbrace{\left(\frac{\cos^3\theta + \sin^3\theta}{\cos^2\theta + \sin^2\theta}\right)}_{\text{bounded}} = \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta) = 0$$

Limit exists.

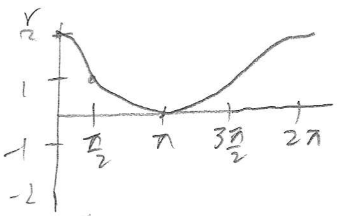
limit = 0

10

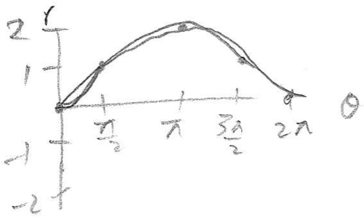
Name: ______________ SID ______________ GSI: ______________

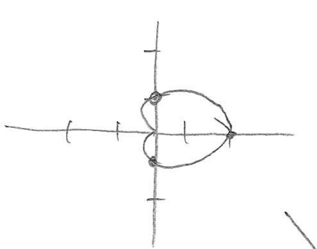
5. (10 points) Find the area of the region that lies inside both of the polar curves $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$. (To evaluate the integral, the identity $\cos^2\theta = (1 + \cos 2\theta)/2$ may be helpful.)

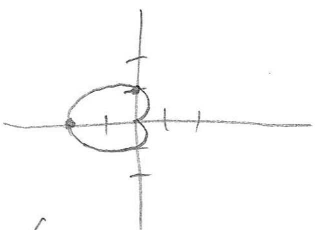
$r = 1 + \cos\theta$



$r = 1 - \cos\theta$





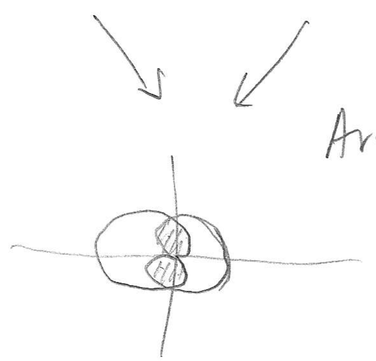


$$1 + \cos\theta = 1 - \cos\theta$$

$$\cos\theta = -\cos\theta$$

$$\theta = \pi/2,\ 3\pi/2$$

By symmetry,



$$\text{Area} = 4\left[\frac{1}{2}\int_0^{\pi/2}(1-\cos\theta)^2\,d\theta\right]$$

$$= 2\int_0^{\pi/2}(1 - 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= 2\int_0^{\pi/2}\left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= 2\int_0^{\pi/2}\left(\frac{3}{2} - 2\cos\theta + \frac{\cos 2\theta}{2}\right)d\theta = 2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2}$$

$$= \left[3\theta - 4\sin\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \left[\left(\frac{3\pi}{2} - 4 + 0\right) - (0 + 0 + 0)\right]$$

$$= \boxed{\frac{3\pi}{2} - 4}$$

tw

Name: ______________________ SID: ______________ GSI: ______________

6. (10 points) Let

$$f(x,y) = e^{x^2+2y^2}.$$

Find the unit vector $\mathbf{u} = \langle a, b\rangle$ which *minimizes* the directional derivative $D_{\mathbf{u}}f$ at the point $(x,y) = (2,3)$.

$D_u f = \nabla f \cdot u$

$= |\nabla f||u|\cos\theta$

$= |\nabla f|\cos\theta$ $\quad -1 \le \cos\theta \le 1$

so $D_u f$ is smallest when $|u|$ is opposite direction of ∇f,

$\theta = \pi$

$\nabla f = \langle (e^{x^2+2y^2})(2x), (e^{x^2+2y^2})(4y)\rangle$ $\quad e^{4+2(9)}$

$= \langle (e^{4+18})(4), e^{22}(12)\rangle$

$\Rightarrow \langle 4e^{22}, 12e^{22}\rangle$

$\Rightarrow \langle 1e^{22}, 3e^{22}\rangle$

$\frac{\nabla f}{4e^{22}} \Rightarrow \langle 1, 3\rangle$

$\frac{-\nabla f}{4e^{22}} = -\langle 1, 3\rangle = \langle -1, -3\rangle$ $\quad \sqrt{1^2+3^2} = \sqrt{10}$

$\boxed{\text{Unit vector} = \langle \frac{-1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}\rangle = \vec{u}}$