University of California, Berkeley, *Department of Physics* *Spring Semester 2010*

Physics 137A

MIDTERM EXAMINATION II April 9, 2010 9:10-10:00 am

Advice:

Please cross out any work which you do not wish to be graded. If your paper is neat, clear, and easy to read, it could affect your grade favorably.

Partial credit will be given for an incomplete or incorrect solution only for relevant, applicable statements that are logically presented. Random, disconnected comments will not be credited even if they happen to be correct. If you are unable to complete the answer to a question, please state clearly how far you got, and indicate how you would proceed to a solution.

Information:

$$\int_{-\infty}^{\infty} e^{-x^2} = \pi^{1/2}$$

$$\int \theta^2 \sin^2\theta d\theta = (\theta^3/6) - (\theta/4)\cos 2\theta - (1/4)(\theta^2 - 1/2)\sin 2\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

1	14
2	10

24/30

14

1. (18 points)

A particle moves along the x-axis in a region of zero potential between the points $|x| = a/2$. The normalized eigenfunction of the first excited state is

$$\psi(x,t) = (2/a)^{1/2} \sin(2\pi x/a) \exp(-iEt/\hbar) \qquad |x| \le a/2$$
$$= 0 \qquad |x| > a/2$$

(a) Find the value of E such that ψ satisfies the time-dependent Schrödinger equation.
(b) Calculate $\langle x^2 \rangle$.
(c) Calculate $\langle p^2 \rangle$.
(d) Calculate the uncertainty product $\Delta x \Delta p$, where $\Delta x = \langle x^2 \rangle^{1/2}$ and $\Delta p = \langle p^2 \rangle^{1/2}$. Show that this product exceeds the minimum uncertainty product $\hbar/2$.

Note the integrals given on the front page!

a. $H\Psi = E\Psi \quad H = \boxed{i\hbar\, \partial/\partial t} = E$

1/5

b. $\langle x^2 \rangle = \int_{-a/2}^{a/2} \psi^* x^2 \psi \, dx = \frac{2}{a} \int_{-a/2}^{a/2} \sin\left(\frac{2\pi x}{a}\right) x^2 \cancel{e^{-iEt/\hbar}} \cancel{e^{iEt/\hbar}} \, dx =$

$= \frac{2}{a}\left(\frac{a}{2\pi}\right)^3 \int_{-a/2}^{a/2} \sin\left(\frac{2\pi x}{a}\right)\left(\frac{2\pi x}{a}\right)^2 d\left(\frac{2\pi x}{a}\right) =$

$\frac{a^2}{2^2\pi^3}\left[\left(\frac{2\pi x}{a}\right)^3 \frac{1}{6} - \left(\frac{2\pi x}{4a}\right)\underbrace{\cos\left(\frac{4\pi x}{a}\right)}_{1} - \left(\frac{1}{4}\right)\left(\frac{4\pi^2 x^2}{a^2} - \frac{1}{2}\right)\sin\left(\frac{4\pi x}{a}\right)\right]_{-a/2}^{a/2}$

$= \frac{a^2}{4\pi^3}\left[\frac{\pi^3}{6}(2) - \frac{\pi}{4}(2)\right] = \boxed{\frac{a^2}{4\pi^3}\left[\frac{\pi^3}{3} - \frac{\pi}{2}\right]}$ 6/6

$\int \sin^2 x = \int \frac{1}{2} - \frac{\cos 2x}{2} = \frac{1}{2}x - \frac{\sin 2x}{4}$

c. $\langle p^2 \rangle = \int_{-a/2}^{a/2} \psi^* (-\hbar^2)\, d^2/dx^2\, \psi \, dx$

$= \frac{2}{a}\int \sin\left(\frac{2\pi x}{a}\right)(-\hbar^2)(-1)\left(\sin\frac{2\pi x}{a}\right)\left(\frac{2\pi}{a}\right)^2 dx$

$= \frac{4\pi}{a^2}\hbar^2 \int \sin^2\left(\frac{2\pi x}{a}\right) d\left(\frac{2\pi x}{a}\right) = \frac{4\pi\hbar^2}{a^2}\left[\frac{\pi x}{a} - \frac{1}{4}\sin\frac{4\pi x}{a}\right]_{-a/2}^{a/2}$

$= \frac{4\pi\hbar^2}{a^2}\left(2 \cdot \frac{\pi}{2}\right) = \boxed{\frac{4\pi^2\hbar^2}{a^2}}$ 5/5

d $\Delta x \Delta p = \frac{a}{2\pi^{3/2}}\left(\frac{\pi^3}{3} - \frac{\pi}{2}\right)^{1/2}\left(\frac{2\pi\hbar}{a}\right) = \frac{\pi\hbar}{\pi^{3/2}}\left(\frac{\pi^3}{3} - \frac{\pi}{2}\right)^{1/2} = \hbar(1.7) > \frac{\hbar}{2}$

2/2

2. (12 points)

10

A particle of mass m is in the state

$$\Psi(x,t) = Ae^{-\alpha[(mx^2/\hbar) + it]},$$

where A and α are real, positive constants, and $i = \sqrt{-1}$.

(a) Find the potential energy V(x) for which $\Psi(x,t)$ satisfies the time-dependent Schrödinger equation.
(b) Find the energy E of the particle in the state $\Psi(x,t)$.
(c) It should now be evident that $\Psi(x,t)$ represents the ground state of a simple harmonic oscillator. From your result in either (a) or (b), express α in terms of the oscillator angular frequency ω.

[Hint: You do not need to find A.]

a. $\Psi = Ae^{-\alpha(mx^2/\hbar + it)} = A \cdot e^{-\alpha mx^2/\hbar} \cdot e^{-\alpha it}$

$\frac{\hbar^2}{2m}\nabla \Psi + V(x,t)\,) = i\hbar \frac{\partial \Psi}{\partial t}$

$-\frac{\hbar^2}{2m}\left(\frac{2x\alpha m}{\hbar}\Psi\right) + \frac{\hbar^2}{2m}\frac{\partial V}{\partial x} = i\hbar(\Psi)\alpha i = -\hbar\alpha\Psi$

3/4

$V = kx^2$ $\quad P = \sqrt{\frac{2k}{m}}$ $\quad \omega = \sqrt{\frac{m}{2k}}$ $\quad k = \frac{m}{2\omega^2}$

$\boxed{V = \frac{2m}{\hbar}\alpha^2 x^2}$

b. $i\hbar\frac{\partial}{\partial t}(\Psi) = \Psi(\alpha i)(i\hbar)(-1)$

$\boxed{E = \alpha\hbar}$ ✓ 4/4

c. $\omega = 2E/\hbar = 2\alpha\hbar/\hbar = \boxed{2\alpha}$

3/4

End