White

Physics 137A/2, Spring 2002

Final Exam

Useful formulae:

$$\sin\theta\cos\phi = \frac{1}{2}\left(\sin[\theta+\phi] + \sin[\theta-\phi]\right) \tag{1}$$

$$\int_{-a}^{a} x\sin bx = 2\frac{\sin ab - ab\cos ab}{b^2} \tag{2}$$

$$\int dx\,\sqrt{a-bx} = \frac{2}{3}(x-\frac{a}{b})\sqrt{a-bx} \tag{3}$$

Gradient in spherical-polar coordinates:

$$\vec{\nabla} = \hat{r}\partial_r + \hat{\theta}r^{-1}\partial_\theta + \hat{\phi}(r\sin\theta)^{-1}\partial_\phi \tag{4}$$

Laplacian in cylindrical coordinates:

$$\nabla^2\psi = \frac{1}{\rho}\partial_\rho\left(\rho\partial_\rho\psi\right) + \frac{1}{\rho^2}\partial_\phi^2\psi + \partial_z^2\psi \tag{5}$$

Bessel's equation (for integer m):

$$x^2y'' + xy' + \left(x^2 - m^2\right)y = 0 \tag{6}$$

with solutions $J_m(x)$ and $Y_m(x)$. Only $J_m(x)$ is regular at the origin.

1. A system can be in one of two states, a ground state $|0\rangle$ of energy 0 and a first excited state $|1\rangle$ of energy ϵ. In the energy eigenbasis a particular observable A has

$$\langle 0|\, A\,|0\rangle \;=\; \langle 1|\, A\,|1\rangle = 0 \tag{7}$$

$$\langle 0|\, A\,|1\rangle \;=\; \langle 1|\, A\,|0\rangle = 1 \tag{8}$$

If the system is prepared in the first excited state by the experimenter, what possible values of A can be measured and with what probabilities? **(5 pnt)**

2. Consider the $n = 2$, $\ell = 1$ state of hydrogen, i.e. $2p$. Give the asymptotic behaviour of the radial wavefunction as $r \to 0$ and $r \to \infty$ in units of the Bohr radius, i.e. with $a_0 = 1$. **(5 pnt)**

3. A particle of mass m is confined in a 1D potential $V(x) = F|x|$, with $-\infty < x < \infty$. Using the WKB approximation, calculate the energy levels E_n for this system. **(10 pnt)**

4. Find the energy eigenvalues and eigenvectors of a 2D circular box with impermeable rigid walls; i.e. a particle lying in the $x - y$ plane is confined by a potential $V(\rho, \phi) = 0$ when $\rho < a$ and $V(\rho, \phi) = \infty$ when $\rho \geq a$. You may express your results in terms of the nth root of the mth Bessel function, $x_{n,m}$, without explicit evaluation.
Hint: Use separation of variables for $\psi(\rho, \phi)$ and change variables to $x = k\rho$ where $\hbar k \equiv \sqrt{2mE}$. **(10 pnt)**

5. A particle of mass m is in a 1D square well potential $V(x) = \infty$ for $|x| \geq a$ and $V(x) = 0$ for $|x| < a$.

(a) Write down the *normalized* eigenfunctions, $\psi_n(x)$, and eigenvalues, E_n, for this system. Determine the parity of the eigenfunctions.

(b) If initially the system is in state

$$\psi(x, t = 0) \propto \psi_1(x) + \psi_2(x) \tag{9}$$

where $n = 1$ is the ground state, what is the state of the system at some later time $t > 0$ (include the normalization constant in your answer)?

(c) A linear potential $\delta V(x) = V_1 x$, with V_1 constant, is added to the Hamiltonian. Calculate the energy of the ground state to first non-trivial order in $|V_1| \ll 1$ using perturbation theory. Is the ground state energy raised or lowered by this perturbation?

(20 pnt)