PHY 137A (D. Budker) Midterm 1 Solutions

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1. Derive the general solution of the Schrödinger Equation for a free particle.

   Solution: The Schrödinger Equation for a free particle is given by

   $$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} = i\hbar\frac{\partial\psi(x)}{\partial t} \tag{1.1}$$

   We use the seperation of variables trick, considering a solution of the form $\psi(x,t) = \psi(x)\chi(t)$. Then, as in the general case, this reduces the time dependent Schrödinger equation to the time independent Schrödinger Equation for every positive value of energy E, $\psi_E(x,t) = \psi_E(x)e^{-iEt/\hbar}$. That is, $\psi_E(x)$ is a solution to

   $$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_E(x)}{\partial x^2} = E\psi_E(x) \tag{1.2}$$

   We make the ansatz that the general solution to this equation has the form

   $$\psi_E(x) = Ae^{ikx} + Be^{-ikx}, \tag{1.3}$$

   where $k^2 = \frac{2mE}{\hbar^2}$. Plugging this into the time independent Schrödinger equation, we get that

   $$\begin{aligned} -\frac{\hbar^2}{2m}\frac{\partial^2\psi_E(x)}{\partial x^2} &= -\frac{\hbar^2}{2m}(A(ik)^2e^{ikx} + (-ik)^2Be^{-ikx}) \\ &= \frac{\hbar^2k^2}{2m}(Ae^{ikx} + Be^{-ikx}) = E(Ae^{ikx} + Be^{-ikx}) = E\psi_E(x), \end{aligned} \tag{1.4}$$

   which shows that our ansatz is indeed a solution. That this is the most general solution follows from the fact that a second order linear differential equation has precisely two linearly independent solutions, and we indeed have exactly two free parameters.

   Now, since the time dependent Schrödinger equation is a linear differential equation, a general solution for a free particle is given by some normalizable superposition of the $\psi_E(x,t)$ for possibly all values of E,

   $$\psi(x,t) = \int_0^\infty c(E)\psi_E(x,t)dE \tag{1.5}$$

   It is more convenient to use k instead of E as the variable of integration, so one instead writes the most general solution as

   $$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{ikx - i\frac{\hbar k^2 t}{2m}}dk. \tag{1.6}$$

   where $\phi(k)$ can be determined from initial conditions and the requirement that $\psi(x,t)$ is properly normalized.

2. Capillary waves on the surface of water are due to surface tension. They prevail over gravity waves when the wavelengths are smaller or on the order of $1cm$. For such waves, the frequency ω scales with the wavelength according to:

   $$\omega \propto \lambda^{-3/2} \tag{1.7}$$

   (a) How does the phase velocity scale with wavelength for such waves?

   (b) How does the group velocity scale with wavelength?

(c) Find the ratio of the group and phase velocities. Explain the meaning of your result.

Solution: We can use the relation between wavelength and wave vectors, $k = \frac{2\pi}{\lambda}$ to rewrite the dispersion relation as

$$\omega = Ck^{\frac{3}{2}}, \tag{1.8}$$

The phase velocity and group velocity are given by the expressions

$$\text{Phase Velocity} = \frac{\omega}{k} = Ck^{\frac{1}{2}} \propto \lambda^{-\frac{1}{2}} \tag{1.9}$$

$$\text{Group Velocity} = \frac{\partial \omega}{\partial k} = \frac{3}{2}Ck^{\frac{1}{2}} \propto \lambda^{-\frac{1}{2}}. \tag{1.10}$$

Thus, the ratio of the group velocity to the phase velocity is clearly $\frac{3}{2}$. This means that the group of waves forming a particular packet is moving faster than any individual crest of the wave. So, if we were far away, we would see the swell of the wave moving past us rather quickly, but each crest would seem to lag behind the group, falling further and further behind till it died away.

(d) Why does this question appear on a quantum mechanics test?

TA's take: Well, understanding the behaviour of waves is after all one of the best ways to gain intuition for the wave like behaviour of quantum particles, and nonrelativistic particles have a dispersion relation not too different from the above relation $\omega \propto \lambda^{-2}$. In particular, the qualitative behaviour of matter waves is much the same as what you've found for capillary waves, but with some altered constants.

3. Consider the scattering of a particle with $E \gg \frac{m\alpha}{2\hbar^2}$, from a triple δ-function potential:

$$V(x) = \alpha \left[\delta(x-a) + \delta(x-b) + \delta(x-c)\right] \tag{1.11}$$

(a) What are the relative positions of the potential spikes (a, b, c) that maximize the reflection coefficient.

Solution: The amplitude for reflection off any given well is given by

$$M = \frac{i\beta}{1 - i\beta}, \tag{1.12}$$

where $\beta = \frac{m\alpha^2}{E\hbar^2}$. Now, in the limit that $E \gg \frac{m\alpha^2}{2\hbar^2}$, $\beta \ll 1$, and thus, we see that the reflection amplitudes are very small, and we can approximate the reflection amplitude for scattering off all three delta functions as the sum of the amplitudes for scattering off each delta function seperately, along with a phase factor which accounts for the extra distance travelled by the corresponding waves. In particular, assuming that the potential spikes are in alphabetical order from left to right, the amplitude for reflection is given by

$$M_3 = M + Me^{2ik(b-a)} + Me^{2ik(c-a)}. \tag{1.13}$$

We will get maximal constructive interference if the reflected waves off all three potential spikes all have the same phase, which happens if we satisfy the conditions

$$2k(b-a) = 2\pi n \tag{1.14}$$

$$2k(c-a) = 2\pi m. \tag{1.15}$$

This will happen, for example, if $(b-a) = \pi/k$ and $(c-a) = 2\pi/k$. In terms of the DeBroigle wavelength of the particle $\lambda = \frac{2\pi}{k}$, this just translates into the requirement that the potential spikes each be seperated by half integral multiples of its DeBroigle wavelength.

(b) How does the reflection coefficient in the arrangement of part (a) compare to the reflection coefficient from a single δ-function potential?

Solution: In this case, $M_3 = 3M$ so $R = |M_3|^2 = 9|M|^2$, is nine times what we would have found for the single delta function potential.

4. Consider a particle of mass m confined in a potential of the form $V(x) = a \cdot x^4$.

   (a) Sketch the relative arrangement of energy levels. Discuss whether the interval between adjacent levels is uniform, and if not, does it increase or decrease for higher energy levels?

   Solution: This potential can be thought of as being something between the harmonic oscillator and the infinite square well. The energy levels of the harmonic oscillator $E_n = (n + \frac{1}{2})\hbar\omega$ are evenly spaced, while the energy levels of the infinite square well, $E_n = \frac{n^2\hbar^2\pi^2}{8ma^2}$ have an increasing interval between subsequent energy levels. Thus, we expect that the energy levels of the potential $V(x) = a \cdot x^4$ should not be evenly spaced, but instead have an increasing spacing which is, nevertheless, at a slower pace than that of the infinite square well.

   (b) Up to a numerical factor, what is the energy of the lowest state.

   Solution: There are many ways in which one could go about approximating the energy of the lowest state. For example, the energy of the lowest state can be well approximated by using the uncertainty principle and the classical Virial Theorem. Let us suppose that the total classical energy of the bound state is E. Then, classically, the particle can be thought of as bound to a region where $V(x) < E$. In particular, the size of this region is just twice the distance from the origin to the classical turning point. The classical turning point is found by solving

$$V(x_c) = a \cdot x_c^4 = E \tag{1.16}$$

   for x_c, which gives us $x_c = (E/a)^{1/4}$. Thus, the particle can be thought of as being confined to a region of size $2x_c$, which can be interpreted quantum mechanically as approximately its uncertainty in position, $\Delta x = 2x_c = 2(E/a)^{1/4}$. Now, the uncertainty principle tells us that this means that its uncertainty in momentum must obey the bound

$$\Delta x \Delta p = 2(E/a)^{1/4} \Delta p \geq \frac{\hbar}{2}. \tag{1.17}$$

   Now, since the particle is bound, we expect its average momentum to be zero, so we would expect that $\Delta p^2 = \langle p^2 \rangle - \langle p \rangle^2 = \langle p^2 \rangle$. This means that $\Delta p^2/2m = \left\langle \frac{p^2}{2m} \right\rangle$ is the contribution of the kinetic energy to the total energy of the particle. But how much of the total energy for this system is coming from kinetic energy? Well, the classical Virial Theorem, tells us that for a particle moving in a bounded orbit in a potential of the form $V(x) = a \cdot x^n$, the average value of the kinetic and potential energies are related by:

$$2\left\langle \frac{p^2}{2m} \right\rangle = n \langle V(x) \rangle \tag{1.18}$$

   This relation also holds in quantum mechanics, where the average values are now interpreted as expectation values for eigenstates of the Hamiltonian. In particular, for $n = 4$, we see that the average value of the potential energy is half that of the kinetic energy. Thus, we expect that for the ground state

$$\frac{\Delta p^2}{2m} = \frac{2E}{3} \tag{1.19}$$

   Plugging this expression into the uncertainty principle

$$\Delta x \Delta p = 2(E/a)^{1/4} \left(\frac{4mE}{3} \right)^{1/2} \geq \frac{\hbar}{2}. \tag{1.20}$$

   we find that

$$E \approx \left(\frac{\sqrt{3}\hbar a^{1/4}}{8m^{1/2}} \right)^{4/3}. \tag{1.21}$$

(c) Calculate the numerical values of this energy (in eV)assuming that the particle is an electron, and $V(1\mathring{A}) = 10eV$ ($1\mathring{A} = 10^{-8}$ cm).

Solution: Since the particle is an electron, we use the fact that its mass is $9.11 \times 10^{-31}kg$. Then, evaluating the energy from above we get

$$E = \left(\frac{\sqrt{3}\hbar a^{1/4}}{8m^{1/2}}\right)^{4/3} = \left(\frac{\sqrt{3} \cdot 6.63 \times 10^{-34} J \cdot s \cdot (1.6 \times 10^{-18})^{1/4}}{16\pi \cdot 10^{-10} m \cdot (9.11 \times 10^{-31} kg)^{1/2}}\right)^{4/3} /1.6 \times 10^{-19} eV/J = 1.09eV \tag{1.22}$$