Name (Last, First): ______________________________

Student ID: ______________________________

Circle your section:

201	Shin	8am	71 Evans	212	Lim	1pm	3105 Etcheverry
202	Cho	8am	75 Evans	213	Tanzer	2pm	35 Evans
203	Shin	9am	105 Latimer	214	Moody	2pm	81 Evans
204	Cho	9am	254 Sutardja Dai	215	Tanzer	3pm	206 Wheeler
205	Zhou	10am	254 Sutardja Dai	216	Moody	3pm	61 Evans
206	Theerakarn	10am	179 Stanley	217	Lim	8am	310 Hearst
207	Theerakarn	11am	179 Stanley	218	Moody	5pm	71 Evans
208	Zhou	11am	254 Sutardja Dai	219	Lee	5pm	3111 Etcheverry
209	Wong	12pm	3 Evans	220	Williams	12pm	289 Cory
210	Tabrizian	12pm	9 Evans	221	Williams	3pm	140 Barrows
211	Wong	1pm	254 Sutardja Dai	222	Williams	2pm	220 Wheeler

If none of the above, please explain: ______________________________

This is a closed book exam, no notes allowed. It consists of 6 problems, each worth 10 points, of which you must complete 5. **Choose one problem not to be graded by crossing it out in the box below.** If you forget to cross out a problem, we will roll a die to choose one for you.

Problem	Maximum Score	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total Possible	50	

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Problem 1) Decide if the following statements are ALWAYS TRUE or SOMETIMES FALSE. You do not need to justify your answers. Enter your answers of **T** or **F** in the boxes of the chart. (Correct answers receive 2 points, incorrect answers -2 points, blank answers 0 points.)

Statement	1	2	3	4	5
Answer	T	F	T	F	F

1) If a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is given by a matrix A, then the range of T is equal to the column space of A.

Both subspaces are the span of the columns of A.

2) If two matrices have equal reduced row echelon forms, then their column spaces are equal.

Counterexample:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

3) If a finite set of vectors spans a vector space, then some subset of the vectors is a basis.

If the set is linearly independent, then it is a basis. If not, some vector is a linear combination of the others. Throw out that vector and check that remaining set still spans. Repeat until set is linearly independent.

4) If A is a 2×2 matrix such that $A^2 = 0$, then $A = 0$.

Counterexample:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

5) If A is a 5×5 matrix such that $\det(2A) = 2\det(A)$, then $A = 0$.

Counterexample:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Problem 2) Indicate with an **X** in the chart all of the answers that satisfy the questions below. You do not need to justify your answers. It is possible that any number of the answers (including possibly none) satisfy the questions. (A completely correct row of the chart receives 2 points, a partially correct row receives 1 point, but any incorrect X in a row leads to 0 points.)

	(*a*)	(*b*)	(*c*)	(*d*)	(*e*)
Question 1		X			
Question 2		X			X
Question 3	X	X	X		
Question 4	X	X		X	
Question 5			X	X	X

Inside of $\mathbb{R}^3$, consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_6 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

1) Which of the following lists are linearly independent?

a) $\mathbf{v}_1, \mathbf{v}_2$.
b) $\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$.
c) $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5$.
d) $\mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_6$.
e) $\mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6$.

2) Which of the following lists span $\mathbb{R}^3$?

a) $\mathbf{v}_1, \mathbf{v}_2$.
b) $\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$.
c) $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5$.
d) $\mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_6$.
e) $\mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6$.

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3) Which of the following matrices have reduced row echelon form equal to $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$?

$a)\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}\quad b)\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}\quad c)\ \begin{bmatrix} 2 & 2 & 2 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\quad d)\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}\quad e)\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

4) Inside of $\mathbb{R}^3$, consider the subset of vectors

$$\{\mathbf{v} = \begin{bmatrix} a \\ b \\ a \end{bmatrix}\}$$

satisfying the following requirements. Which of them are subspaces?

a) a and b are both zero.
b) a is any number and b is zero.
c) a is zero or b is zero or both are zero.
d) a and b are equal.
e) a, b are both positive, both negative, or both zero.

5) Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ has 2-dimensional range and we know

$$T(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \qquad T(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which of the following are a possible standard matrix of T?

$a)\ \begin{bmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ 1 & -1 & 0 \end{bmatrix}\quad b)\ \begin{bmatrix} 1 & 1 & -2 \\ 2 & 2 & -4 \\ 1 & -1 & -2 \end{bmatrix}\quad c)\ \begin{bmatrix} 1 & 1 & -2 \\ 0 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix}\quad d)\ \begin{bmatrix} 1 & 1 & -2 \\ 0 & 2 & -2 \\ 2 & -1 & -1 \end{bmatrix}\quad e)\ \begin{bmatrix} -1 & 1 & 0 \\ 0 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix}$

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Problem 3) For a real number c, consider the linear system

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & cx_3 & + & x_4 & = & c \\ & & -x_2 & + & x_3 & + & 2x_4 & = & 0 \\ x_1 & + & 2x_2 & + & x_3 & - & x_4 & = & -c \end{array}$$

a) (5 points) For what c, does the linear system have a solution?

Let us find the REF of the augmented matrix

$$\begin{bmatrix} 1 & 1 & c & 1 & | & c \\ 0 & -1 & 1 & 2 & | & 0 \\ 1 & 2 & 1 & -1 & | & -c \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & c & 1 & | & c \\ 0 & -1 & 1 & 2 & | & 0 \\ 0 & 1 & 1-c & -2 & | & -2c \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & c & 1 & | & c \\ 0 & -1 & 1 & 2 & | & 0 \\ 0 & 0 & 2-c & 0 & | & -2c \end{bmatrix}$$

Thus the linear system has a solution if and only if $c \neq 2$.

b) (5 points) Find a basis of the subspace of solutions when $c = 0$.

When $c = 0$, the REF of the unaugmented matrix is

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

The free variable is x_4 and so solutions are of the form

$$\begin{bmatrix} -3x_4 \\ 2x_4 \\ 0 \\ x_4 \end{bmatrix}$$

Thus a basis consists of the single vector

$$\begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

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Problem 4) (10 points) Let $\mathbb{P}_2$ be the vector space of polynomials of degree less than or equal to 2. Let B be the basis $\mathbf{b}_1 = x^2, \mathbf{b}_2 = -1 + x, \mathbf{b}_3 = x + x^2$.

Find the coordinates of the vector $\mathbf{v} = 1 + 2x - x^2$ with respect to B.

Writing all polynomials in terms of the standard basis $1, x, x^2$, we find we must solve the linear system with augmented matrix

$$\begin{bmatrix} 0 & -1 & 0 & | & 1 \\ 0 & 1 & 1 & | & 2 \\ 1 & 0 & 1 & | & -1 \end{bmatrix}$$

Let us put it into REF

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & | & -1 \\ 0 & 1 & 1 & | & 2 \\ 0 & -1 & 0 & | & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & | & -1 \\ 0 & 1 & 1 & | & 2 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$$

Now we find the solution which is the sought-after coordinate vector

$$[\mathbf{v}]_B = \begin{bmatrix} -4 \\ -1 \\ 3 \end{bmatrix}$$

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Problem 5) Consider the matrices

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & -1 \\ 0 & -1 & 2 & -1 \end{bmatrix}$$

1) (5 points) Calculate the matrix AB.

$$AB = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 4 & 1 & 2 & -1 \\ 2 & 1 & 0 & -1 \\ 1 & -1 & 3 & -3 \end{bmatrix}$$

2) (5 points) Calculate the determinant $\det(AB)$. Cite any methods used in your answer.

$\det(AB) = 0$ since AB is not invertible. The reason AB is not invertible is AB has a nontrivial null space. The reason AB has a nontrivial null space is that B maps $\mathbb{R}^4$ to $\mathbb{R}^3$ and so B must have a nontrivial null space, and AB results from first applying B then A, so any vector in the null space of B will be in the null space of AB.

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Problem 6)

1) (6 points) Fill in the blanks (each worth 1/2 a point) in the proof of the following assertion.

Assertion. If A is a square matrix, and the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is injective, then the linear transformation $\mathbf{x} \mapsto A^T\mathbf{x}$ is injective.

Proof. For any $m \times n$ matrix A, recall that

$$n = \underline{rank(A)} + \underline{\dim Nul(A)}$$

and similarly for A^T, we have

$$m = \underline{rank(A^T)} + \underline{\dim Nul(A^T)}$$

We also know for A and A^T that

$$\underline{rank(A)} = \underline{rank(A^T)}$$

Next recall that $\mathbf{x} \mapsto A\mathbf{x}$ is injective if and only if

$$\underline{\dim Nul(A)} = 0$$

and similarly, $\mathbf{x} \mapsto A^T\mathbf{x}$ is injective if and only if

$$\underline{\dim Nul(A^T)} = 0$$

Thus when A is square, so $m = n$, and $\mathbf{x} \mapsto A\mathbf{x}$ is injective, we have

$$\underline{rank(A)} = n = m = \underline{rank(A^T)} + \underline{\dim Nul(A^T)}$$

And so we conclude that

$$\underline{\dim Nul(A^T)} = 0$$

and hence $\mathbf{x} \mapsto A^T\mathbf{x}$ is injective.

2) (4 points) Give an example of a 2×2 matrix A such that $Nul(A) \neq Nul(A^T)$.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad Nul(A) = Span\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$$

$$A^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad Nul(A^T) = Span\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$$