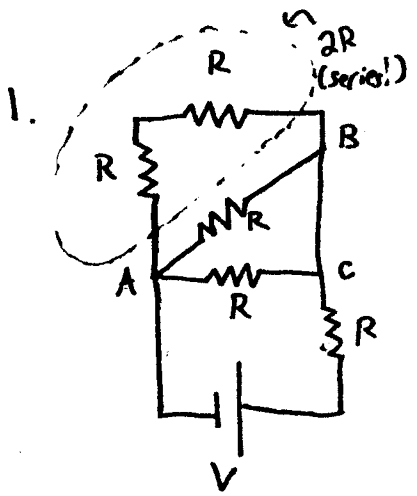
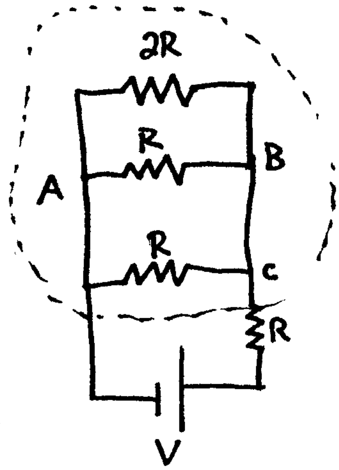
Huang Fall 07 MT2 Solutions

1.



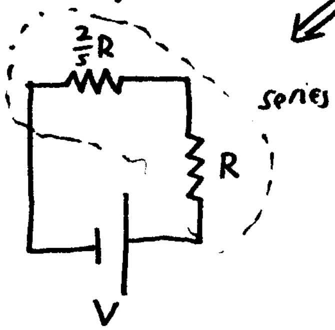
$\Rightarrow$



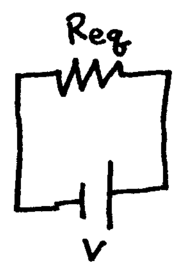
Parallel:

$R_{eq}^{-1} = \frac{1}{R} + \frac{1}{R} + \frac{1}{2R}$

$= \frac{5}{2R} \Rightarrow R_{eq} = \frac{2}{5}R$



$\Rightarrow$



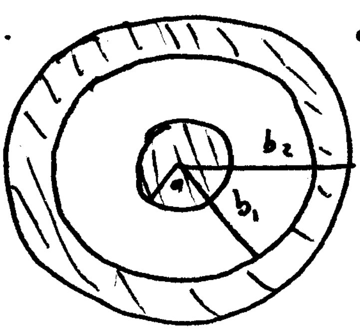
$\Rightarrow R_{eq} = \frac{2}{5}R + R$

$\Rightarrow \boxed{R_{eq} = \frac{7}{5}R}$

$R = 2.8\ k\Omega$

$\Rightarrow \boxed{R_{eq} = 3.92\ k\Omega}$

2.



a) To find C:

- Put charge $+Q$ on inner conductor, $-Q$ on outer.

$\Rightarrow$ Charge $+Q$ @ radius a & $-Q$ at b_1

[all of $-Q$ @ b_1, since $\vec{E} = 0$ inside $b_1 \to b_2$, so $Q_{enc} = 0$ if you put a Gaussian sphere @ radius $b_1 < r < b_2$!]

· Find $\vec{E}$. Spherical Symmetry ⇒ Gauss' Law says $\vec{E} = \frac{Q_{enc}(r)}{4\pi\epsilon_0 r^2}\hat{r}$

$$Q_{enc}(r) = \begin{cases} 0 & a > r \\ Q & b_1 > r > a \\ 0 & r > b_1 \end{cases}$$

$$\Rightarrow \vec{E} = \begin{cases} \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} & a < r < b \\ 0 & \text{else} \end{cases}$$

· Find ΔV: Take a radial path from a to b_1 ⇒ $d\vec{l} = dr\,\hat{r}$

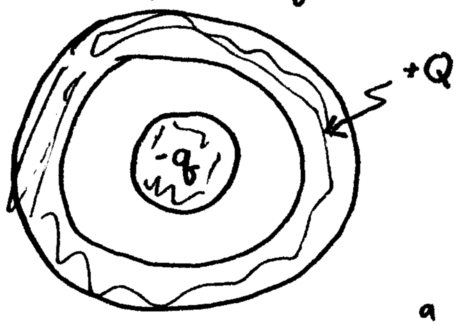
$$\text{ᵢ} \quad \Delta V = -\int_a^{b_1} \vec{E}(r)\cdot(dr\,\hat{r}) = -\int_a^{b_1} \frac{Q}{4\pi\epsilon_0 r^2}\,dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{b_1} - \frac{1}{a}\right)$$

$$\cdot \quad C = \frac{|Q|}{|V|} = \frac{Q}{\frac{Q}{4\pi\epsilon_0}\left|\frac{1}{b_1} - \frac{1}{a}\right|}$$

$$b_1 > a \Rightarrow \frac{1}{b_1} < \frac{1}{a} \Rightarrow \left|\frac{1}{b_1} - \frac{1}{a}\right| = \frac{1}{a} - \frac{1}{b_1} = \frac{b_1 - a}{ab_1}$$

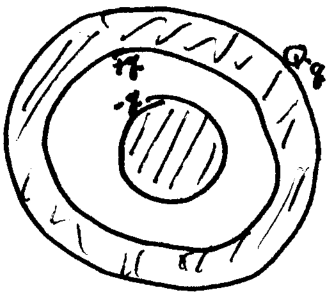
$$\Rightarrow \boxed{C = \frac{4\pi\epsilon_0 a b_1}{b_1 - a}}$$

Now, put $-q$ on small cond. & $+Q$ on large.



$r < a$ & $b_1 < r < b_2$ are <u>conductors</u>, so $\vec{E} = 0$

Putting a Gaussian sphere @ $b_1 < r < b_2$ gives a flux of 0, so $q_{enc} = 0$ ⇒ a charge $+q$ is on surface b_1 & $Q - q$ on surface b_2.



$$\Rightarrow Q_{enc}(r) = \begin{cases} 0 & r < a \\ -q & a < r < b_1 \\ 0 & b_1 < r < b_2 \\ Q - q & b_2 < r \end{cases}$$

⇒ b) $r<a \Rightarrow \boxed{\vec{E}=0}$

c) $a<r<b_1 \Rightarrow \boxed{\vec{E}=\frac{-q}{4\pi\epsilon_0 r^2}\hat{r}}$

d) $b_1<r<b_2 \Rightarrow \boxed{\vec{E}=0}$

e) $b_2<r \Rightarrow \boxed{\vec{E}=\frac{Q-q}{4\pi\epsilon_0 r^2}\hat{r}}$

f) $-q$ on $r=a$
$+q$ on surface $r=b_1$
$Q-q$ on surface $r=b_2$,
all evenly distributed

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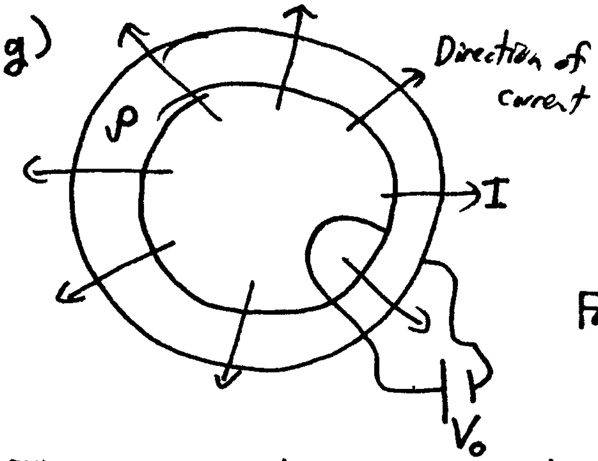
g)



$V_0 = RI$

?

For constant cross-sectional area,
$R=\frac{\rho l}{A}$

The cross-section here varies, though. Take a very thin slice of the sphere @ rad. r & width dr. Then, $l = dr$ & the area is about constant, with $A(r)=4\pi r^2$

$$\Rightarrow dR = \frac{\rho\, dr}{4\pi r^2}$$

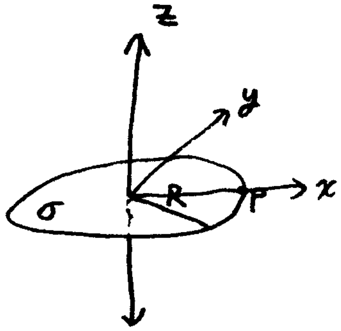
resistance of piece

These are connected in series from $r=a$ to $r=b$, so $R = \sum dR = \int_a^b \frac{\rho\, dr}{4\pi r^2}$

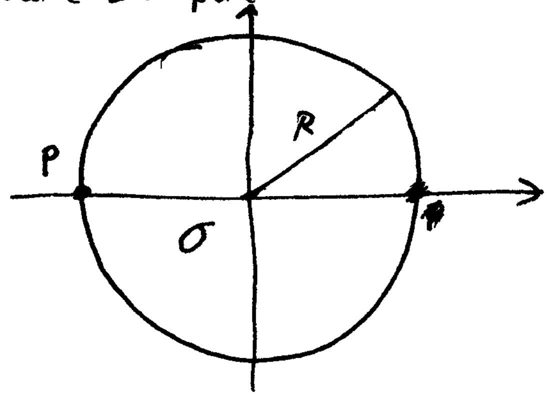
$$= \frac{\rho}{4\pi}\left.\frac{-1}{r}\right|_a^b \Rightarrow R = \frac{\rho}{4\pi}\frac{b-a}{ab}$$

$$\Rightarrow \boxed{I = \frac{4\pi ab V_0}{\rho(b-a)}}$$

3.

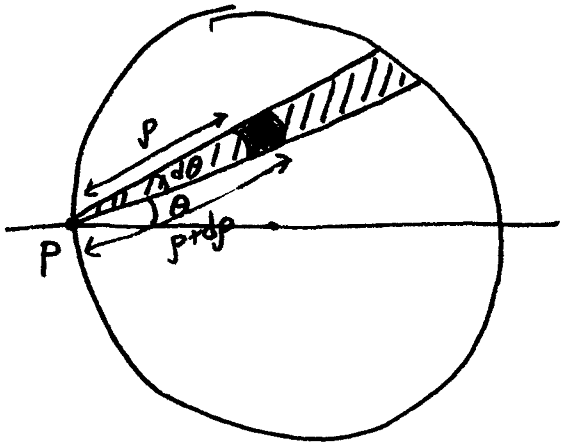


Look @ $z=0$ plane:



Set $V=0$ @ $\infty \Rightarrow V$ can be found by adding the potentials from individual charges.

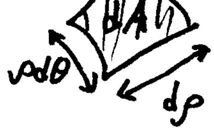
First, suppose we had a "slice" of the disk extending out from point p, such that the width of the slice went through an angle $d\theta$ & the slice is about @ angle θ:



Let the distance from P be called ρ & consider a small piece of our slice going from ρ to $\rho+d\rho$.

The area of this slice is now small - it has an appx. length of $d\rho$ & appx. width of $\rho d\theta$, so

$$dA = \rho\, d\rho\, d\theta$$



The charge on this piece is $dq = \sigma dA = \sigma \rho\, d\rho\, d\theta$ & the potential @ P due to this piece is

$$dV = \frac{dq}{4\pi\epsilon_0 \rho} = \frac{\sigma \cancel{\rho}\, d\rho\, d\theta}{4\pi\epsilon_0 \cancel{\rho}}$$

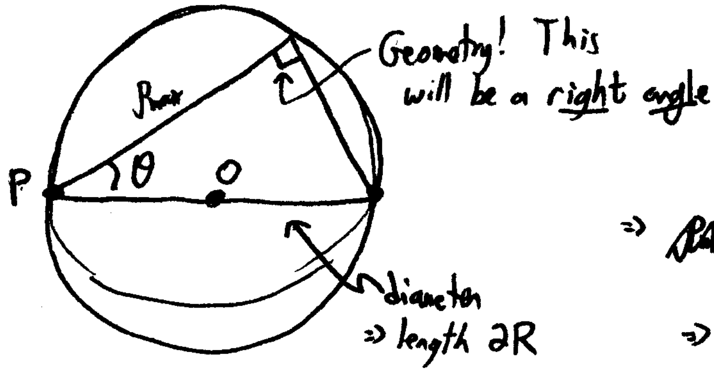
$$\Rightarrow dV = \frac{\sigma\, d\rho\, d\theta}{4\pi\epsilon_0}$$

To get the full potential @ P, we merely need to add these up!
[superposition!]

First, add up the pieces into the slice

$$dV = \int_{\rho=0}^{\rho=?} \frac{\sigma \, d\rho \, d\theta}{4\pi\epsilon_0}$$

We obviously start @ $\rho = 0$, but what is ρ_{max}? It will depend on θ:



$$\Rightarrow \cancel{\rho_{max}} \cdot \cos\theta = \frac{\rho_{max}}{2R}$$

$$\Rightarrow \rho_{max} = 2R\cos\theta$$

$$\Rightarrow dV = \int_0^{2R\cos\theta} \frac{\sigma \, d\rho \, d\theta}{4\pi\epsilon_0} = \frac{\cancel{2}\sigma R \cos\theta \, d\theta}{2\cancel{4}\pi\epsilon_0}$$

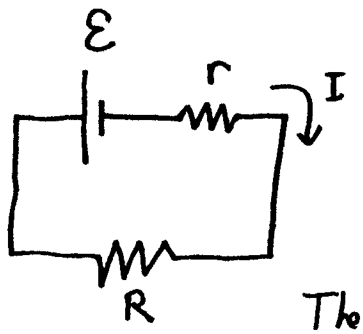
$$\Rightarrow dV_{slice} = \frac{\sigma R}{2\pi\epsilon_0} \cos\theta \, d\theta$$

Now, add the slices: we want to start with the slice "straight down" from p to the one "straight up" so we get the full disk.

$$\Rightarrow \theta \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

$$V = \frac{\sigma R}{2\pi\epsilon_0} \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \, d\theta}_{2} \quad \Rightarrow \boxed{V = \frac{\sigma R}{\pi\epsilon_0}}$$

4.



• Net resistance is $r+R$, so current is

$$I = \frac{\mathcal{E}}{R+r}$$

The power dissipated by R is:

$$P = I^2 R = \frac{\mathcal{E}^2 R}{(R+r)^2}$$

Keeping $\mathcal{E}$ & r fixed, we want to find the value of R s.t. P is a max.

$$\Rightarrow \frac{dP}{dR} = 0 \quad \Rightarrow \quad 0 = \frac{(R+r)^{\cancel{2}}\mathcal{E}^2 - \mathcal{E}^2 R\, 2\cancel{(R+r)}}{(R+r)^{\cancel{4}\,3}} \cancel{= 0}$$

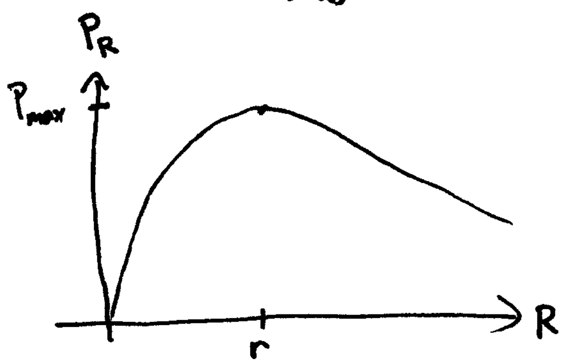
$$\Rightarrow \mathcal{E}^2\left[(R+r) - 2R\right] = 0 \quad \Rightarrow \quad r - R = 0 \quad \Rightarrow \quad \boxed{R = r}$$

Is this a max? $\cancel{\frac{d^2P}{dR^2} = (R+r)^3\left[\mathcal{E}^2 - 2\mathcal{E}^2 2R - 2r\mathcal{E}^2\right]}$

$$\cancel{\frac{d^2P}{dR^2}} \qquad \frac{dP}{dR} = \mathcal{E}^2 \frac{r-R}{(r+R)^3}$$

$$\Rightarrow \frac{d^2P}{dR^2} = \mathcal{E}^2 \frac{(r+R)^{\cancel{3}}(-1) - (r-R)3\cancel{(r+R)^2}}{(r+R)^{\cancel{6}\,4}} = \frac{-\mathcal{E}^2}{(r+R)^4}\left(r+R+3r-3R\right)$$

$$= -\frac{2\mathcal{E}^2}{(r+R)^4}(2r-R) \quad \Rightarrow \quad \left.\frac{d^2P}{dR^2}\right|_{R=r} = \frac{-2\mathcal{E}^2}{(2r)^4}\, r < 0 \Rightarrow \text{max} \checkmark$$



$$P_{max} = \frac{\mathcal{E}^2 r}{(2r)^2} = \frac{\mathcal{E}^2}{4r}$$