Problem 1

There are two ways to approach this problem. Method number 1 involves finding the total force on both point charges and then taking expansions. The electric field due to the line charge is given on the equation sheet:

$$\vec{E}(\vec{r}) = \frac{\lambda}{2\pi\epsilon_0|\vec{r}|}\hat{r} = \frac{\lambda}{2\pi\epsilon_0|\vec{r}|^2}\vec{r}$$

(Note that r in this equation is NOT the r in the question)
So the total force is the sum of the force on the positive and negative charge. Let's look at the $\theta = 0$ case first, since in this case $\hat{r} = \hat{x}$ and everything looks nice:

$$\begin{aligned}
\vec{F}(\theta = 0) &= q\vec{E}(\vec{r}_{+q}) - q\vec{E}(\vec{r}_{-q}) \\
&= \frac{q\lambda}{2\pi\epsilon_0}\left(\frac{1}{r+\frac{d}{2}} - \frac{1}{r-\frac{d}{2}}\right)\hat{x} \\
&= \frac{q\lambda}{2\pi\epsilon_0 r}\left(\left(1+\frac{d}{2r}\right)^{-1} - \left(1-\frac{d}{2r}\right)^{-1}\right)\hat{x} \\
&\approx \frac{q\lambda}{2\pi\epsilon_0 r}\left(1-\frac{d}{2r} - \left(1+\frac{d}{2r}\right)\right)\hat{x} \\
&= -\frac{qd\lambda}{2\pi\epsilon_0 r^2}\hat{x}
\end{aligned}$$

Here, since $\frac{d}{2r}$ is small, I am able to use the binomial expansion $(1+x)^{-1} \approx 1 - x$ given on the equation sheet.
Now consider the $\theta = 90°$ case. Now $\vec{r}$ has two components. Using the Pythagorean theorem, I get that:

$$\vec{r}_{+q} = \sqrt{r^2 - \left(\frac{d}{2}\right)^2}\,\hat{x} + \frac{d}{2}\hat{y}$$

$$\vec{r}_{-q} = \sqrt{r^2 - \left(\frac{d}{2}\right)^2}\,\hat{x} - \frac{d}{2}\hat{y}$$

so the total force is:

$$\begin{aligned}
\vec{F}(\theta = 90°) &= q\vec{E}(\vec{r}_{+q}) - q\vec{E}(\vec{r}_{-q}) \\
&= \frac{q\lambda}{2\pi\epsilon_0 r^2}\left(\sqrt{r^2 - \left(\frac{d}{2}\right)^2}\,\hat{x} + \frac{d}{2}\hat{y} - \left(\sqrt{r^2 - \left(\frac{d}{2}\right)^2}\,\hat{x} - \frac{d}{2}\hat{y}\right)\right) \\
&= \frac{qd\lambda}{2\pi\epsilon_0 r^2}\hat{y}
\end{aligned}$$

The other way to do this was to think in terms of energy. We can approximate the physical dipole with an ideal dipole with $|\vec{p}| = qd$. Let us suppose that the dipole sits at some point (x, y). Then:

$$U = -\vec{p}\cdot\vec{E} = -\frac{qd\lambda}{2\pi\epsilon_0(x^2+y^2)}(\hat{p}\cdot(x\hat{x} + y\hat{y}))$$

Where $\hat{p}$ is the unit vector of the direction the dipole points. When $\theta = 0$, $\hat{p} = \hat{x}$, so:

$$U(\theta = 0) = -\frac{qd\lambda x}{2\pi\epsilon_0(x^2+y^2)}$$

And:

$$\vec{F}(\theta=0) = -\vec{\nabla}U = \frac{qd\lambda}{2\pi\epsilon_0}\left(\frac{y^2-x^2}{(x^2+y^2)^2}\hat{x} - \frac{2xy}{(x^2+y^2)^2}\hat{y}\right)$$

Now I evaluate this at $x=r$ and $y=0$, as that is where the dipole is:

$$\vec{F}(\theta=0) = -\frac{qd\lambda}{2\pi\epsilon_0 r^2}\hat{x}$$

When $\theta = 90°$, $\hat{p} = \hat{y}$, so:

$$U(\theta=90°) = -\frac{qd\lambda y}{2\pi\epsilon_0(x^2+y^2)}$$

$$\vec{F}(\theta=90°) = -\vec{\nabla}U = \frac{qd\lambda}{2\pi\epsilon_0}\left(-\frac{2xy}{(x^2+y^2)^2}\hat{x} + \frac{x^2-y^2}{(x^2+y^2)^2}\hat{y}\right)$$

Again, evaluate at $x=r$ and $y=0$:

$$\vec{F}(\theta=90°) = \frac{qd\lambda}{2\pi\epsilon_0 r^2}\hat{y}$$

Problem 2

Part a

Consider a cylindrical Gaussian surface with its axis in the x direction, and the two caps (with area A) at $x < 0$ and $x > d$ respectively. The electric fluxes through the top and bottom caps are zero, since the E–field in these two regions are zero. By the translation symmetry of the charge distribution in the y and z directions, we know that $\vec{E} = E(x)\hat{x}$. Therefore, $\Phi_{\text{side}} = \int \vec{E} \cdot \hat{n} da = 0$, since $\hat{n}$ is perpendicular to $\vec{E}$. Putting these together, the total flux through the Gaussian surface is zero. By Gauss's Law, the enclosed charge is necessarily zero. This means

$$\begin{aligned}
0 &= A \int_0^d \rho(x) dx \\
0 &= A\rho_0 \frac{d}{n\pi} \left(\sin\left(\frac{n\pi d}{d}\right) - \sin\left(\frac{n\pi 0}{d}\right) \right) \qquad (\text{for } n \neq 0) \\
0 &= \sin(n\pi),
\end{aligned}$$

which is satisfied when n is a non-zero integer.

Part b

This time, consider a cylindrical Gaussian surface with its axis in the x direction, and the two caps (with area A) at $x < 0$ and $x = h$ respectively, where $0 < h < d$. Using the arguement above, $\Phi_{\text{bottom}} = \Phi_{\text{side}} = 0$, and $\Phi_{\text{top}} = AE(h)$. The enclosed charge is given by

$$\begin{aligned}
q_{\text{inc}} &= A \int_0^h \rho(x) dx \\
&= A\rho_0 \frac{d}{n\pi} \sin\left(\frac{n\pi h}{d}\right).
\end{aligned}$$

By Gauss's Law, therefore,

$$\begin{aligned}
AE(h) &= A\frac{\rho_0}{\epsilon_0} \frac{d}{n\pi} \sin\left(\frac{n\pi h}{d}\right) \\
\vec{E}(x) &= \frac{\rho_0}{\epsilon_0} \frac{d}{n\pi} \sin\left(\frac{n\pi x}{d}\right) \hat{x}.
\end{aligned}$$

Using a line integral from $x = 0$ to $x = g$, we can then obtain the potential at $x = g$.

$$
\begin{aligned}
\phi(g) &= -\int_0^g \vec{E}(x) \cdot (dx\,\hat{x}) \\
&= -\int_0^g \frac{\rho_0}{\epsilon_0} \frac{d}{n\pi} \sin\left(\frac{n\pi x}{d}\right) dx \\
&= \frac{\rho_0}{\epsilon_0} \frac{d^2}{n^2\pi^2} \left(\cos\left(\frac{n\pi g}{d}\right) - \cos\left(\frac{n\pi 0}{d}\right)\right) \\
&= \frac{\rho_0}{\epsilon_0} \frac{d^2}{n^2\pi^2} \left(\cos\left(\frac{n\pi g}{d} - 1\right)\right) \\
\phi(x) &= \frac{\rho_0}{\epsilon_0} \frac{d^2}{n^2\pi^2} \left(\cos\left(\frac{n\pi x}{d} - 1\right)\right) && \text{for } 0 \leq x \leq d \\
&= 0 && \text{for } x < 0 \\
&= \frac{\rho_0}{\epsilon_0} \frac{d^2}{n^2\pi^2} (\cos(n\pi) - 1) && \text{for } x > d.
\end{aligned}
$$

Problem 3

Consider building this charge distribution shell by shell, from the inside out. At a certain step, the radius that is already built is b ($b \leq a$), and we bring in an infinitesimal amount of charge to build the next shell with thickness Δb and area $4\pi b^2$. The amount of charge moved is then $\Delta q = 4\pi b^2 \Delta b \rho$. This movement of charge happens under the E–field from the already–built sphere. To find this E–field, we consider a gaussian surface with radius $d \geq b$. The charge enclosed is $4/3\pi b^3 \rho + Q$. Due to the spherical symmetry of the charge distribution, the E-field must be in the form $\vec{E} = E(r)\hat{r}$, which means $\Phi = 4\pi d^2 E(d)$. By Gauss's Law, we thus have

$$
\begin{aligned}
4\pi d^2 E(d) &= \frac{4/3\pi b^3 \rho + Q}{\epsilon_0} \\
E(d) &= \frac{4/3\pi b^3 \rho + Q}{4\pi\epsilon_0 d^2} \\
\vec{E}(r) &= \frac{4/3\pi b^3 \rho + Q}{4\pi\epsilon_0 r^2}\hat{r}.
\end{aligned}
$$

The work required on our part to move the infinitesimal charge from infinity to b is therefore

$$
\begin{aligned}
\Delta W &= \int \vec{F}_{\text{by us}} \cdot \vec{dl} \\
&= \int_b^{\infty} (-\vec{E}(r)\Delta q) \cdot (dr\,\hat{r}) \\
&= -\Delta q \int_b^{\infty} \frac{4/3\pi b^3\rho + Q}{4\pi\epsilon_0 r^2} dr \\
&= -\Delta q \frac{4/3\pi b^3\rho + Q}{4\pi\epsilon_0} \int_b^{\infty} \frac{1}{r^2} dr \\
&= \Delta q \frac{4/3\pi b^3\rho + Q}{4\pi\epsilon_0} \frac{1}{b} \\
&= 4\pi b^2 \Delta b \rho \frac{4/3\pi b^3\rho + Q}{4\pi\epsilon_0} \frac{1}{b} \\
&= b \Delta b \rho \frac{4/3\pi b^3\rho + Q}{\epsilon_0}.
\end{aligned}
$$

The total work required on our part to construct the complete distribution would simply be the sum of all these ΔW. In the limit of Δb going to zero, this sum becomes the integral

$$
\begin{aligned}
W &= \int_0^a b\,db\,\rho \frac{4/3\pi b^3\rho + Q}{\epsilon_0} \\
&= \frac{\rho}{\epsilon_0} \int_0^a db \left(\frac{4}{3}\pi b^4 \rho + Qb\right) \\
&= \frac{\rho}{\epsilon_0} \left(\frac{4}{15}\pi a^5 \rho + \frac{1}{2} Q a^2\right).
\end{aligned}
$$

If this turns out to be zero,

$$
Q = -\frac{8}{15}\pi a^3 \rho.
$$

Problem 4

a)

The left hand side of the capacitor plate will be positively charged (as it is attached to the positive end of the battery). Since (positive) current is leaving it, and because current is $\frac{dQ}{dt}$, this means that charge is leaving that plate. Since $Q = CV_S$, and V_S cannot be changing here since it is always connected to the battery, this means that C must be decreasing.
Now let us determine the equivalent capacitance of the capacitor. This is just two capacitors in parallel:

$$C_{eq} = \frac{\epsilon_0 s}{d}(\kappa y + (s - y))$$

For this to decrease, y, the height of the dielectric, must decrease.

b)

The current leaving the capacitor is:

$$I = \frac{dQ}{dt} = \frac{dC_{eq}}{dt}V_S = V_S \frac{\epsilon_0 s}{d}(\kappa - 1)\frac{dy}{dt}$$

Then:

$$\frac{dV}{dt} = sd\frac{dy}{dt} = \frac{Id^2}{V_S \epsilon_0 (\kappa - 1)}$$

#5

This problem can be done by super position. To calculate the sphere's field take a spherical Gaussian surface. By Symmetry the field should point radially, so $\int E \cdot da = 4\pi r^2 E = Q/\epsilon_0$. Q will only be non-zero when the sphere is taken with radius larger then L, in which case $Q = \sigma 4\pi L^2$. Thus, the electric field due to the sphere is:

$$E_{sph} = \begin{cases} 0 & r < L \\ \frac{k(4\pi L^2)\sigma_S}{r^2}\hat{r} & r > L \end{cases}$$

By spherical symmetry we also know that the sphere can be treated as a point charge. The field of the plane on the z-axis is given by summing the fields of many infinite lines. These fields are $E_{line} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}$ by Gauss' Law. To see this, take a cylindrical Gaussian surface of radius r, length l around the infinite line, by symmetry the field points radially and thus only has flux through the lateral surface. Then, $\int E \cdot da = 2\pi r l E = Q/\epsilon_0 = \lambda l/\epsilon_0$, rearranging yields the result.

Here we have a planar density so, $\lambda \to \sigma$ giving te field per unit length across the strip. We know that the field will ultimately point along the z direction so it is useful to take that component when we sum them up,

$$E_{plane}(z) = \int_{-L}^{L} \left(\frac{\sigma_L}{2\pi\epsilon_0\sqrt{x^2+z^2}} \right) \frac{z}{\sqrt{x^2+z^2}} \hat{z} dx = \frac{\sigma_L}{2\pi\epsilon_0} 2\tan^{-1}\left(\frac{L}{z}\right)\hat{z}$$

Thus we get at total field,

$$E = \begin{cases} \frac{\sigma_L}{2\pi\epsilon_0} 2\tan^{-1}\left(\frac{L}{z}\right)\hat{z} & |z| < L \\ \frac{\sigma_L}{2\pi\epsilon_0} 2\tan^{-1}\left(\frac{L}{z}\right)\hat{z} + \frac{z}{|z|}\frac{L^2\sigma_S}{\epsilon_0 z^2}\hat{z} & |z| > L \end{cases}$$