

Physics 7C Fall 2014 Midterm 2 Solutions

All problems are worth 10 points. To receive full credit you must show **all** of your work, however, you do not need to derive any formulas that appear on this exam, or that you can recall from memory.

- (1) As outlaws escape in their getaway car, which goes $\frac{3}{4}c$, a police officer fires a bullet from the pursuit car behind them, which only goes $\frac{1}{2}c$. The muzzle velocity of the bullet (relative to the gun) is $\frac{1}{3}c$. Does the bullet reach the outlaws,
 - (a) according to Newtonian Physics?
 - (b) according to Special Relativity?
- (2) Suppose that our Sun is about to explode. In an effort to escape, we depart in a spaceship at $\frac{3}{5}c$ and head toward the star Tau Ceti (at rest with respect to the Sun) a distance L away, at time $t = t' = 0$ in the rest frames of the Sun and the ship respectively. When we reach the midpoint of our journey we see our Sun explode (the light from the explosion reaches the ship) and, unfortunately, at the same instant we see Tau Ceti explode as well (the light from the explosion reaches the ship).
 - (a) In the frame of reference in which the Sun and Tau Ceti are at rest, did they explode simultaneously? If so, how long after we started our journey did they explode? If not, what is the time difference between the two explosions?
 - (b) In the spaceship's frame of reference should we conclude that the two explosions occurred simultaneously? If so, how long after we started our journey did they explode? If not, what is the time difference between the two explosions?
- (3) Consider the two events: (x_1, t_1) and (x_2, t_2) . Define $\Delta x \equiv x_2 - x_1$ and $\Delta t \equiv t_2 - t_1$.
 - (a) If the events are spacelike separated, find the velocity of the frame of reference in which the two events happen at the same time. What is the distance between these two events in this reference frame? Write your answers in terms of Δx and Δt .
 - (b) If the events are timelike separated, find the velocity of the frame of reference in which the two events happen at the same place. What is the time difference between these two events in this reference frame? Write your answers in terms of Δx and Δt .
- (4) Suppose you have a collection of particles, all moving along the x-axis, with energies, E_1, E_2, \dots, E_n and momenta p_1, p_2, \dots, p_n . The total momentum is the sum of the individual momenta and the total energy is the sum of the individual energies.
 - (a) Find the velocity, v_{cm} , of the frame of reference in which the total momentum is zero.
 - (b) Show that $|v_{\text{cm}}| \leq c$.
- (5) We wish to create a particle of mass M by smashing together two particles of mass $m < M$. It turns out that it is much more efficient to smash the particles together with equal and opposite momenta than with one of them at rest. What is the total initial kinetic energy of the two colliding particles of mass m , if
 - (a) they are moving toward each other with equal and opposite momenta?
 - (b) one of them is at rest?
- (6) A particle of charge 1C and mass 1kg is at rest at the origin at time $t = 0$, and in the presence of a uniform E-field, $\mathbf{E} = E_0 \hat{\mathbf{x}}$. You may assume that the particle only moves along the x-axis (which is true). In Newtonian Mechanics the velocity of the particle is $u(t) = E_0 t$ and the position is $x(t) = \frac{1}{2} E_0 t^2$. In Relativistic Mechanics,
 - (a) what is the velocity of the particle as a function of time?
 - (b) what is the position of the particle as a function of time?

- (1) The velocity of the bullet is $u = \frac{1}{3}c$ in the rest frame of the pursuit car. The original frame of reference is moving with velocity $v = -\frac{1}{2}c$ with respect to the pursuit car.

- (a) According to Newtonian Physics, the velocity of the bullet in the original frame of reference is

$$u' = u - v = \frac{1}{3}c + \frac{1}{2}c = \frac{5}{6}c > \frac{3}{4}c$$

Thus, the bullet will hit the outlaws.

- (b) According to Special Relativity, the velocity of the bullet in the original frame of reference is

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}} = \frac{\frac{5}{6}c}{1 - \frac{1}{6}} = \frac{5}{7}c = \frac{20}{28}c < \frac{21}{28}c = \frac{3}{4}c$$

Thus, the bullet will not hit the outlaws.

- (2) (a) In the rest frame of the Sun the ship moves along the trajectory $x = vt = \frac{3}{5}ct$. At time $t = 0$ the Sun is at $x = 0$ and Tau Ceti is at $x = L$. The light from the Sun and Tau Ceti must both travel a distance $\frac{L}{2}$ to reach the midpoint, which takes $\frac{L}{2c}$ seconds. If t_1 is the time at which the Sun explodes, and t_2 is the time at which Tau Ceti explodes, the light from the Sun reaches the midpoint at time $t_1 + \frac{L}{2c}$ and the light from Tau Ceti reaches the midpoint at time $t_2 + \frac{L}{2c}$. The ship is at the midpoint when $t = \frac{L}{2v}$. Clearly these two times both equal $\frac{L}{2v}$ when

$$t_1 = t_2 = \frac{L}{2v} - \frac{L}{2c} = \frac{L}{3c}$$

- (b) In the rest frame of the ship both stars move to the left with speed $v = \frac{3}{5}c$. At time $t' = 0$ the Sun is at $x' = 0$ and Tau Ceti is at $x' = \frac{L}{\gamma}$. If the Sun explodes at time t'_1 , the light from it must travel a distance vt'_1 to reach the ship, which takes $\frac{vt'_1}{c}$ seconds. If Tau Ceti explodes at time t'_2 , the light must travel a distance $\frac{L}{\gamma} - vt'_2$ to reach the ship, which takes $\frac{L}{\gamma c} - \frac{vt'_2}{c}$ seconds. The ship is halfway between the two planets at time $t' = \frac{L}{2\gamma v}$. Thus,

$$\begin{aligned} \frac{2L}{3c} &= \frac{L}{2\gamma v} = t'_1 + \frac{vt'_1}{c} = \frac{8}{5}t'_1 \Rightarrow t'_1 = \frac{5L}{12c} \\ \frac{2L}{3c} &= \frac{L}{2\gamma v} = t'_2 + \frac{L}{\gamma c} - \frac{vt'_2}{c} = \frac{4L}{5c} + \frac{2}{5}t'_2 \Rightarrow t'_2 = -\frac{L}{3c} \end{aligned}$$

Thus $t_1 - t_2 = \frac{3L}{4c}$. We also could have determined this using

$$\Delta t' = \gamma \left(\Delta t - \frac{v}{c^2} \Delta x \right) = \frac{5}{4} \left(0 - \frac{3}{5c} (-L) \right) = \frac{3L}{4c}$$

- (3) In both cases define $\Delta x \equiv x_2 - x_1$ and $\Delta t = t_2 - t_1$

- (a) If $\Delta s^2 < 0$ then $c\Delta t < \Delta x$.

$$0 = \Delta t' = \gamma \left(\Delta t - \frac{v}{c^2} \Delta x \right) \Rightarrow v = \left(\frac{c\Delta t}{\Delta x} \right) c < c$$

The spacial distance between the two events in this frame is the proper distance

$$\Delta x' = L_p = \sqrt{-\Delta s^2} = \sqrt{\Delta x^2 - c^2 \Delta t^2}$$

- (b) If $\Delta s^2 > 0$ then $c\Delta t > \Delta x$.

$$0 = \Delta x' = \gamma (\Delta x - v\Delta t) \Rightarrow v = \left(\frac{\Delta x}{c\Delta t} \right) c < c$$

The time difference between the two events in this frame is the proper time

$$\Delta t' = \Delta \tau = \frac{\Delta s}{c} = \frac{\sqrt{c^2 \Delta t^2 - \Delta x^2}}{c}$$

(4) The total momentum and energy are

$$p = \sum_{i=1}^n p_i, \quad E = \sum_{i=1}^n E_i$$

(a) We wish to go to frame of reference in which $p' = 0$. Thus,

$$0 = p' = \gamma \left(p - \frac{v_{\text{cm}}}{c^2} E \right) \Rightarrow v_{\text{cm}} = \left(\frac{pc}{E} \right) c = \left(\frac{\sum_{i=1}^n p_i c}{\sum_{i=1}^n E_i} \right) c$$

(b) Since $E_i = \sqrt{m_i^2 c^4 + p_i^2 c^2} \geq |p_i| c$,

$$E = \sum_{i=1}^n E_i \geq \sum_{i=1}^n |p_i| c \geq \sum_{i=1}^n p_i c = p \Rightarrow |v_{\text{cm}}| = \left| \frac{pc}{E} \right| c \leq c$$

(5) Let the momentum of the final particle of mass M be q .

(a) Conservation of momentum and energy tells us that

$$\begin{aligned} q &= p - p = 0 \\ E_k + 2mc^2 &= \sqrt{M^2 c^4 + q^2 c^2} = Mc^2 \Rightarrow E_k = Mc^2 - 2mc^2 \end{aligned}$$

(b) Conservation of energy and momentum tells us that

$$\begin{aligned} q &= p \\ E + mc^2 &= \sqrt{M^2 c^4 + q^2 c^2} = \sqrt{M^2 c^4 + p^2 c^2} = \sqrt{M^2 c^4 + E^2 - m^2 c^4} \\ E^2 + 2mc^2 E + m^2 c^4 &= M^2 c^4 + E^2 - m^2 c^4 \\ E &= \frac{M^2}{2m} c^2 - mc^2 \Rightarrow E_k = E - mc^2 = \frac{M^2}{2m} c^2 - 2mc^2 \end{aligned}$$

(6) (a) From the Lorentz Force Law we have

$$E_0 = \frac{dp}{dt} \Rightarrow E_0 t = p = \gamma u \Rightarrow E_0^2 t^2 = \gamma^2 u^2 = \frac{u^2}{1 - \frac{u^2}{c^2}} \Rightarrow u(t) = \frac{E_0 t}{\sqrt{1 + \frac{E_0^2 t^2}{c^2}}}$$

Notice that $u(t) \rightarrow c$ as $t \rightarrow \infty$.

(b) Integrating $u(t)$ we find that

$$x(t) = \int_0^t u(t') dt' = \int_0^t \frac{E_0 t'}{\sqrt{1 + \frac{E_0^2 t'^2}{c^2}}} dt' = \frac{c^2}{E_0} \left[\sqrt{1 + \frac{E_0^2 t'^2}{c^2}} - 1 \right]$$

where the integration was done using a simple u substitution.