ME 106 Fluid Mechanics: Midterm 2
Fall 2014
Name & Discussion Section:

1. Given the unsteady flow field $u = t^2$ and $v = 1 - t$,

   (a) Determine the equation $y(x)$ describing the streamline passing through point $x = 0$ and $y = 0$ at time $t = 2$.

   Equation for streamline at $t = 2$

   $$\left.\frac{dy}{dx}\right|_{t=2} = \left.\frac{v}{u}\right|_{t=2} = \frac{-1}{4} \quad \Rightarrow \quad y(x) = -0.25x + C$$

   Plugging in point $(0, 0)$ we get $0 = 0 + C$ and therefore $C = 0$. Hence,

   $$y(x) = -0.25x$$

   6 points total. 2 pts correct diff eq for streamline + 2 pts correctly solve diff eq + 2 pts correctly solve for C.

   (b) Determine the equations $x(t)$ and $y(t)$ for a particle path passing through point $x = 0$ and $y = 0$ at time $t = 2$.

   For $x(t)$,

   $$\dot{x} = u = t^2 \quad \Rightarrow \quad x(t) = \frac{t^3}{3} + C_x$$

   Since $x(2) = \frac{2^3}{3} + C_x = 0$, we get $C_x = -\frac{8}{3}$.

   For $y(t)$,

   $$\dot{y} = u = 1 - t \quad \Rightarrow \quad y(t) = t - \frac{t^2}{2} + C_y$$

   Since $y(2) = 2 - \frac{2^2}{2} + C_y = 0$, we get $C_y = 0$. Hence,

   $$x(t) = \frac{t^3}{3} - \frac{8}{3}$$
   $$y(t) = -\frac{t^2}{2}$$

   8 points total. 2 pts correct solve for generic $x(t)$ + 2 pts correctly solve for C_x + 2 pts correct solve for generic $y(t)$ + 2 pts correctly solve for C_y.

2. It is known that a shear flow has the velocity profile

$$\begin{aligned} u &= ay + by^2 \\ v &= 0 \\ w &= 0 \end{aligned}$$

Using Navier-Stokes equations, derive the pressure gradient, ∇p, assuming gravity $\vec{g}$ is oriented in negative y-direction.

Plugging this velocity field into Navier-Stokes equations, we see that all terms on left hand side are zero, and on the right hand side we see that the only non-zero term is

$$\mu \frac{\partial^2 u}{\partial y^2} = 2\mu b \, .$$

Therefore, Navier-Stokes equations reduce to

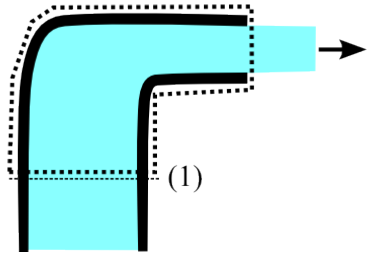
$$\begin{aligned} 0 &= -\frac{\partial p}{\partial x} + 2\mu b \\ 0 &= -\frac{\partial p}{\partial y} - \rho g \\ 0 &= -\frac{\partial p}{\partial z} \end{aligned}$$

And hence

$$\nabla p = \begin{bmatrix} 2\mu b \\ -\rho g \\ 0 \end{bmatrix}$$

9 points total. 3 pts correctly compute material derivative + 3 pts correctly compute viscous term $\mu \Delta \vec{v}$ + 3 points final answer.

3. Consider the steady flow of an incompressible fluid through a bent nozzle that exits to atmospheric conditions. Over the distance shown, viscous effects are negligible. **You are given A_1, v_1, A_e, ρ** (area and speed of the fluid at section 1, the exit area, and fluid density).



(a) Solve for the gauge pressure p_1 in terms of given variables.

Since flow is inviscid, we can use Bernoulli equation. Ignoring gravity effects,

$$p_1 + \frac{1}{2}\rho v_1^2 = \frac{1}{2}\rho v_e^2 \quad \Rightarrow \quad p_1 = \frac{1}{2}\rho(v_e^2 - v_1^2)$$

Based on conservation of mass, $\rho A_1 v_1 = \rho A_e v_e$ and hence $v_e = \frac{A_1}{A_e} v_1$. Therefore

$$p_1 = \frac{1}{2}\rho v_1^2 \left(\frac{A_1^2}{A_e^2} - 1\right)$$

8 points total. 3 pts correct Bernoulli equation (consistent with assumptions) + 3 pts correct conservation of mass + 2 pts final answer.

(b) Solve for the x- and y-components of the anchoring force to hold the nozzle in terms of variables given. (Define control volume.)

Define CV to include fluid and nozzle as shown by dotted line above. Ignore weight of nozzle. Define F_x and F_y as anchoring force components, each assumed to be in positive coordinate direction.

Use momentum equation (steady flow)

$$\Sigma F = \int_{CS} \rho \mathbf{v}(\mathbf{v} \cdot \hat{n}) dA$$

In x-direction,

$$F_x = \rho v_e^2 A_e \qquad \text{(where } v_e \text{ defined above)}$$

In y-direction,

$$F_y + p_1 A_1 = \rho v_1(-v_1)A_1 \quad \Rightarrow \quad F_y = -p_1 A_1 - \rho v_1^2 A_1$$

where p_1 defined above.

10 points total. 2 pts control volume + 3 pts correct x-momentum eqn + 3 pts correct y-momentum eqn + 2 pts final answers.

4. The streamfunction for a "sine-sine" flow is given by $\Psi = \sin(\pi x)\sin(\pi y)$.

(a) Determine the velocity field $u(x, y)$, $v(x, y)$.

$$u = \frac{\partial \Phi}{\partial y} = \pi \sin(\pi x)\cos(\pi y)$$
$$v = -\frac{\partial \Phi}{\partial x} = -\pi \cos(\pi x)\sin(\pi y)$$

4 points total. 2 pts correct u + 2 pts correct v.

(b) Show whether the flow is irrotational.

We need to determine if rotation $\vec{\omega}$ (or vorticity $\vec{\zeta} = 2\vec{\omega} = \nabla \times \vec{v}$) is everywhere zero.

$$\begin{aligned}\vec{\zeta} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \\ &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\hat{k} \\ &= \pi^2 \sin(\pi x)\sin(\pi y) + \pi^2 \sin(\pi x)\sin(\pi y) \\ &\neq 0\end{aligned}$$

Hence flow is <u>not</u> irrotational.

6 points total. 3 pts for stating that irrotational implies $\nabla \times \vec{v} = 0$ + 3 pts for correctly computing $\nabla \times \vec{v}$.

(c) Compute the acceleration of a fluid particle located at $x = 1$ and $y = 0.5$.

$$\begin{aligned}\vec{a} &= \frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} \\ &= u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\begin{bmatrix} u \\ v \\ w \end{bmatrix} \\ &= 0\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} + 0\frac{\partial}{\partial z}\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} \quad \text{(Note } w = 0\text{. And at } x = 1,\ y = 0.5 \text{ we have } u = v = 0.) \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\end{aligned}$$

Alternatively, one could directly note that $(1, 0.5)$ is a "fixed point", i.e., $\vec{v} = \vec{0}$ and hence $\vec{a} = \vec{0}$.

6 points total. 2 pts for correct equation for material derivative + 4 points for correctly computing convective acceleration.

Chapter Summary Equations:

Equation for streamlines	$\frac{dy}{dx} = \frac{v}{u}$
Acceleration	$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}$
Material derivative	$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + (\mathbf{V} \cdot \nabla)(\ )$
Streamwise and normal components of acceleration	$a_s = V\frac{\partial V}{\partial s}, \qquad a_n = \frac{V^2}{\mathcal{R}}$
Reynolds transport theorem (restricted form)	$\frac{DB_{\text{sys}}}{Dt} = \frac{\partial B_{\text{cv}}}{\partial t} + \rho_2 A_2 V_2 b_2 - \rho_1 A_1 V_1 b_1$
Reynolds transport theorem (general form)	$\frac{DB_{\text{sys}}}{Dt} = \frac{\partial}{\partial t}\int_{\text{cv}} \rho b\, d\forall + \int_{\text{cs}} \rho b\, \mathbf{V} \cdot \hat{\mathbf{n}}\, dA$
Relative and absolute velocities	$\mathbf{V} = \mathbf{W} + \mathbf{V}_{\text{cv}}$
Conservation of mass	$\frac{\partial}{\partial t}\int_{\text{cv}} \rho\, d\forall + \int_{\text{cs}} \rho \mathbf{V} \cdot \hat{\mathbf{n}}\, dA = 0$
Mass flowrate	$\dot{m} = \rho Q = \rho A V$
Average velocity	$\overline{V} = \frac{\int_A \rho \mathbf{V} \cdot \hat{\mathbf{n}}\, dA}{\rho A}$
Steady flow mass conservation	$\sum \dot{m}_{\text{out}} - \sum \dot{m}_{\text{in}} = 0$
Moving control volume mass conservation	$\frac{\partial}{\partial t}\int_{\text{cv}} \rho\, d\forall + \int_{\text{cs}} \rho \mathbf{W} \cdot \hat{\mathbf{n}}\, dA = 0$
Deforming control volume mass conservation	$\frac{DM_{\text{sys}}}{Dt} = \frac{\partial}{\partial t}\int_{\text{cv}} \rho\, d\forall + \int_{\text{cs}} \rho \mathbf{W} \cdot \hat{\mathbf{n}}\, dA = 0$
Force related to change in linear momentum	$\frac{\partial}{\partial t}\int_{\text{cv}} \mathbf{V}\rho\, d\forall + \int_{\text{cs}} \mathbf{V}\rho \mathbf{V} \cdot \hat{\mathbf{n}}\, dA = \sum \mathbf{F}_{\substack{\text{contents of the}\\ \text{control volume}}}$
Moving control volume force related to change in linear momentum	$\int_{\text{cs}} \mathbf{W}\rho \mathbf{W} \cdot \hat{\mathbf{n}}\, dA = \sum \mathbf{F}_{\substack{\text{contents of the}\\ \text{control volume}}}$
Vector addition of absolute and relative velocities	$\mathbf{V} = \mathbf{W} + \mathbf{U}$
Acceleration of fluid particle	$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}$
Vorticity	$\zeta = 2\,\boldsymbol{\omega} = \nabla \times \mathbf{V}$
Conservation of mass	$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$
Stream function	$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$

The Navier–Stokes equations

(x direction)

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

(y direction)

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

(z direction)

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$