Avg = 80.82
STDV = 12.389

Name Key

SID

BioE 102 Midterm 2

1. **Pressure Vessel – Cylinder (30 Points)**

Here we'll test the mechanical properties of a large diameter artery (10mm inner diameter; 500μm wall thickness). *Approximate as thin walled cylinder*

a. You perform a tensile test on a piece of this artery and find that a force of 5N causes failure due to shear stress in the $z\theta$ plane. Determine the shear stress required for failure.
b. Knowing the failure shear stress, what is the lowest blood pressure (in units of Pa) is required for failure of the native vessel due to shear in the $z\theta$ plane? (*assume there is no axial load on the vessel*)
c. If graft is applied which applies a force of 1N to the vasculature, what is now the lowest blood pressure that will result in failure of the artery?

a $\sigma_{zz} = \frac{F_f}{2\pi a h}$ $\quad \sigma_{\theta\theta} = 0 \quad \sigma_{z\theta} = 0$

$$\tau_m = \pm\sqrt{\left(\frac{\sigma_{zz}-\sigma_{\theta\theta}}{2}\right)^2 + \sigma_{z\theta}^2} = \pm\frac{\sigma_{zz}}{2} = \pm\frac{F_f}{4\pi a h} = \frac{\pm 5N}{4\pi(5E\text{-}3m)(0.5E\text{-}3m)} = \pm 159\text{ kPa}$$

b. $\sigma_{zz} = \frac{Pa}{2h} \quad \sigma_{\theta\theta} = \frac{Pa}{h} \quad \sigma_{z\theta} = 0$

$$\tau_m = \frac{\pm F_f}{4\pi a h} = \pm\sqrt{\left(\frac{\sigma_{zz}-\sigma_{\theta\theta}}{2}\right)^2 + \sigma_{z\theta}^2} = \pm\frac{Pa}{4h}$$

$$P = \frac{F_f}{\pi a^2} = \frac{5N}{\pi(5E\text{-}3m)^2} = 63.7\text{ kPa}$$

c. $\sigma_{zz} = \frac{Pa}{2h} + \frac{F_g}{2\pi a h} \quad \sigma_{\theta\theta} = \frac{Pa}{h} \quad \sigma_{z\theta} = 0$

$$\tau_m = \frac{\pm F_f}{4\pi a h} = \pm\sqrt{\left(\frac{\sigma_{zz}-\sigma_{\theta\theta}}{2}\right)^2 + \sigma_{z\theta}^2} = \pm\left(\frac{-Pa}{4h} + \frac{F_g}{4\pi a h}\right)$$

$$\begin{cases} P = \frac{F_f - F_g}{\pi a^2} = \frac{5N - 1N}{\pi(5E\text{-}3m)^2} = -51.9\text{ kPa} \\ P = \frac{F_f + F_g}{\pi a^2} = \frac{5N + 1N}{\pi(5E\text{-}3m)^2} = 76.39\text{ kPa} \end{cases}$$

Name Key

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2. Pressure Vessel – Sphere (30 Points)

A cell is placed in a low salt solution resulting in an osmotic gradient that swells the cell with excess fluid. Modelling the cell as a thin walled sphere with an internal radius of $20\mu m$ and a membrane thickness of $10nm$. *Use small deformation assumptions*

a. If the internal pressure rises to 0.3Pa what normal stresses are produced within the cell membrane?
b. Calculate the normal strains provided with a Young's modulus and Poisson's ratio of 5kPa and 0.4.
c. Use these strains to calculate a change in area for a small free body diagram. If any small area of cell membrane in the $\theta\phi$ plane expands by more than 5% without transporting more lipids to the lipid bilayer, the cell will rupture. Is this pressure enough to cause rupture in the cell?

a. $\sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{Pa}{2h} = \frac{0.3\,Pa \cdot 20E\text{-}6m}{2 \cdot 10E\text{-}9m} = 300\ Pa$

$\sigma_{rr} \cong \frac{-P}{2} = \frac{-0.3}{2} = -0.15\ Pa$

b. $\varepsilon_{\theta\theta} = \varepsilon_{\phi\phi} = \frac{1}{E}\left(\sigma_{\theta\theta} - \nu\left(\sigma_{\phi\phi} + \sigma_{rr}\right)\right)$

$= \frac{1}{5E3\,Pa}\left(300\,Pa - 0.4\left(300\,Pa - 0.15\,Pa\right)\right) = 0.036$

$\varepsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - \nu\left(\sigma_{\theta\theta} + \sigma_{\phi\phi}\right)\right)$

$= \frac{1}{5E3\,Pa}\left(-0.15\,Pa - 0.4\left(300\,Pa + 300\,Pa\right)\right) = -0.048$

c $\Delta A_{\theta\phi} = A_f - A_i \cong (1+\varepsilon_{\theta\theta})(1+\varepsilon_{\phi\phi})L^2 - L^2$

$\frac{\Delta A_{\theta\phi}}{A_{\theta\phi}} = \frac{(1+\varepsilon_{\theta\theta})^2 L^2 - L^2}{L^2} = (1+\varepsilon_{\theta\theta})^2 - 1 = 0.073 > 5\%$

$\Rightarrow$ yes, The cell will rupture.

Name Key
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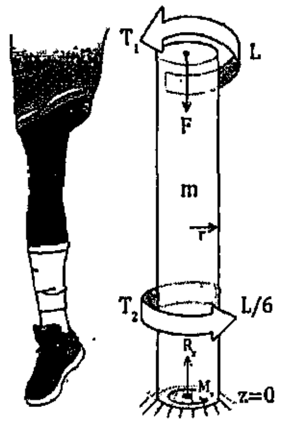
3. Elongation and Torsion (40 Points)

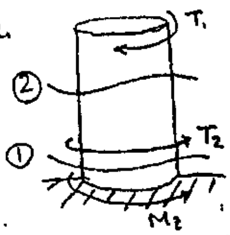
A basketball player lands after taking a rebound while shifting their weight causing extreme torsion and force to their right leg. A force of ~~70~~75kg is weighing down on the top of their femur. The leg has a mass of ~~15~~k 20 distributed evenly throughout the leg. We will model the leg as a cylinder as shown below.

a. Solve for the distribution of force and torque as a function of ~~x~~ z.
b. Describe the regions where the maximum magnitude σ_{zz} and $\sigma_{z\theta}$ exist in terms of (r, z, θ), and solve for their values in those regions.
c. Determine the maximum magnitude of normal stress in the $z\theta$ plane, describe the range where this exists and describe or depict the stress transformation.

T_1	50Nm
T_2	15Nm
m	20kg
F	$\left(75kg \times 9.81\frac{m}{s^2}\right)N$
r	10cm
L	1.2m



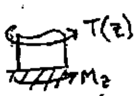
a.



Whole beam

$\Sigma M_z = 0 = M_z + T_2 - T_1 \qquad M_z = T_1 - T_2$

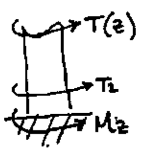
1



$\Sigma M_z = M_z + T(z)$

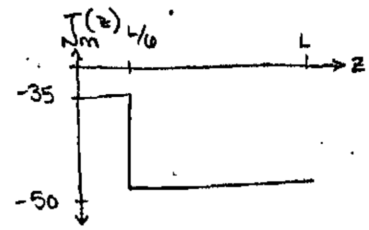
$T(z) = -M_z \qquad 0 < z \le \frac{L}{6}$

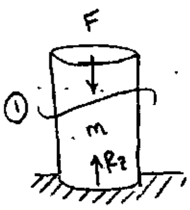
2



$\Sigma M_z = M_z + T_2 + T(z)$

$T(z) = -M_z - T_2 = -T_1 \qquad \frac{L}{6} < z \le L$



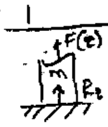


Whole beam

$\Sigma F_z = 0 = R_z - mg - F$

$R_z = mg + F$

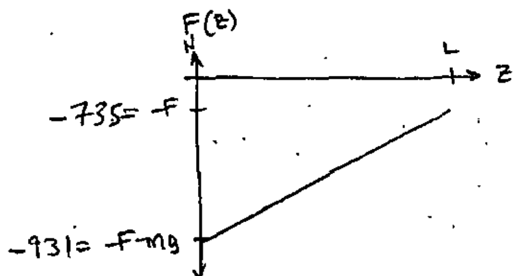
1



$\Sigma F_z = R_z - m\frac{z}{L}g + F(z)$

$F(z) = m\frac{z}{L}g - R_z$

$= mg\frac{z}{L} - mg - F = mg\left(\frac{z-L}{L}\right) - F = 196\left(\frac{z-L}{L}\right) - 735 \quad N$



b. $\sigma_{zz} = \frac{F(z)}{A}$ $\because$ max when $|F(z)|$ is max $\Rightarrow$ max @ $r \in [0, 10cm]$, $z = 0$, $\theta \in \mathbb{R}^s$

$= -29.6\ kPa$

$\sigma_{z\theta} = \frac{T(z)\,r}{J} = \frac{T(z)}{\frac{\pi}{2}r^3}$ $\because$ max when r is max & $|T(z)|$ is max $\Rightarrow$ max @ r = 10cm, $z \in [\frac{L}{6}, L]$, $\theta \in \mathbb{R}^s$

$= -31.83\ kPa$

c. Possible maximums at z = 0 (max F(z)) or $z = \frac{L}{6}$ (max F(z) in region of max. T(z))

$\sigma_{zz} = \frac{F(z)}{A} \qquad \sigma_{z\theta} = \frac{T(z)\,r}{\frac{\pi}{2}r_o^4}$

$\sigma_1 = \frac{\sigma_{zz}}{2} + \sqrt{\left(\frac{\sigma_{zz}}{2}\right)^2 + \sigma_{z\theta}^2}$

@ z=0 $= \frac{-F-mg}{2} - \sqrt{\left(\frac{-F-mg}{2}\right)^2 + \left(\frac{-T_1+T_2}{\frac{\pi}{2}r^3}\right)^2} = -41.6\,kPa$

@ z=L/6 $\frac{-F-\frac{5}{6}mg}{2} - \sqrt{\left(\frac{-F-\frac{5}{6}mg}{2}\right)^2 + \left(\frac{-T_1}{\frac{\pi}{2}r^3}\right)^2} = -49.2\,kPa$

max normal stress of -49.2kPa @ r = 10cm, $z = \frac{L}{6}$, $\theta \in \mathbb{R}^s$

$\alpha_p = \frac{1}{2}\tan^{-1}\left(\frac{2\sigma_{z\theta}}{\sigma_{zz}}\right) = 32.9°$