Math 1B. Solutions to Second Midterm

1. (22 points) The portion of the curve

$$x = 1 + \sqrt{1 - y^2}$$

from $(3/2, \sqrt{3}/2)$ to $(1, 1)$ is rotated about the x-axis. Find the area of the resulting surface.

Solution 1. The derivative dx/dy is

$$\frac{dx}{dy} = \frac{1}{2}(1 - y^2)^{-1/2}(-2y) = -\frac{y}{\sqrt{1 - y^2}} ,$$

so

$$\begin{aligned} ds &= \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \\ &= \sqrt{1 + \left(\frac{-y}{\sqrt{1 - y^2}}\right)^2}\, dy \\ &= \sqrt{1 + \frac{y^2}{1 - y^2}}\, dy \\ &= \sqrt{\frac{1 - y^2 + y^2}{1 - y^2}}\, dy \\ &= \frac{1}{\sqrt{1 - y^2}}\, dy . \end{aligned}$$

Therefore the area is

$$\begin{aligned} \text{Area} &= 2\pi \int_{\sqrt{3}/2}^{1} y\, ds \\ &= 2\pi \int_{\sqrt{3}/2}^{1} \frac{y\, dy}{\sqrt{1 - y^2}} \\ &= 2\pi \left(-\frac{1}{2}\right) \int_{1/4}^{0} \frac{du}{\sqrt{u}} \\ &= -\pi \cdot 2u^{1/2} \Big|_{1/4}^{0} \\ &= -\pi \left(0 - 2\sqrt{\frac{1}{4}}\right) \\ &= \pi . \end{aligned}$$

Here we used the substitution $u = 1 - y^2$, $du = -2\, dy$.

Solution 2. First solve for y in terms of x:

$$\begin{aligned} x-1 &= \sqrt{1-y^2}\ ; \\ x^2-2x+1 &= 1-y^2\ ; \\ y^2 &= 2x-x^2\ ; \\ y &= \sqrt{2x-x^2}\ . \end{aligned}$$

(We take the positive square root because y is given as ranging from $\sqrt{3}/2$ to 1.)

Then the derivative is

$$\frac{dy}{dx} = \frac{1}{2}(2x-x^2)^{-1/2}(2-2x) = \frac{1-x}{\sqrt{2x-x^2}}\ ,$$

and

$$\begin{aligned} ds &= \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= \sqrt{1+\left(\frac{1-x}{\sqrt{2x-x^2}}\right)^2}\,dx \\ &= \sqrt{1+\frac{1-2x+x^2}{2x-x^2}}\,dx \\ &= \sqrt{\frac{2x-x^2+1-2x+x^2}{2x-x^2}}\,dx \\ &= \frac{1}{\sqrt{2x-x^2}}\,dx\ . \end{aligned}$$

Therefore

$$\begin{aligned} \text{Area} &= 2\pi\int_1^{3/2} y\,ds \\ &= 2\pi\int_1^{3/2} \sqrt{2x-x^2}\cdot\frac{dx}{\sqrt{2x-x^2}} \\ &= 2\pi\int_1^{3/2} dx \\ &= 2\pi x\,\Big|_1^{3/2} \\ &= \pi\ . \end{aligned}$$

2. (18 points) *Without using the Comparison Test or Limit Comparison Test,* determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty}\frac{1}{n((\ln n)^2+1)}$$

Use the Integral Test. Note that the functions $(\ln x)^2 + 1$ and x are positive increasing functions of x (for $x \geq 1$), so their product $x((\ln x)^2 + 1)$ is a positive increasing function of x, and therefore the function

$$\frac{1}{x((\ln x)^2 + 1)}$$

is a positive decreasing function of x. It is also continuous. This means that we can apply the Integral Test.

Using the substitution $u = \ln x$, $du = dx/x$, we have

$$\int \frac{dx}{x((\ln x)^2 + 1)} = \int \frac{du}{u^2 + 1} = \arctan u + C = \arctan(\ln x) + C\,.$$

Therefore

$$\begin{aligned}
\int_1^\infty \frac{dx}{x((\ln x)^2 + 1)} &= \lim_{t\to\infty} \int_1^t \frac{dx}{x((\ln x)^2 + 1)} \\
&= \lim_{t\to\infty} \arctan(\ln x)\,\Big|_1^t \\
&= \lim_{t\to\infty} \big(\arctan(\ln t) - 0\big) \\
&= \frac{\pi}{2}\,.
\end{aligned}$$

The last step is true because as $t \to \infty$, $\ln t \to \infty$, and so $\arctan(\ln t) \to \pi/2$.

Since the integral converges, the series also converges. Since all of the terms in the series are positive, the series converges absolutely.

3. (20 points) *Without using l'Hospital's Rule,* find

$$\lim_{x\to 0} \frac{e^{x^2} - (1 + x^2)}{(e^x - 1)^4}$$

From the known series $\sum_{n=0}^\infty x^n/n!$ for e^x, we see that the numerator of the fraction is equal to the power series

$$\sum_{n=2}^\infty \frac{x^{2n}}{n!} = \frac{x^4}{2} + \frac{x^6}{6} + \ldots$$

(note that subtracting off $1 + x^2$ removes the $n = 0$ and $n = 1$ terms from the series). Similarly, the denominator of the fraction is

$$\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)^4 ,$$

and this is a series whose first term is x^4. Cancelling out x^4 from numerator and denominator then gives

$$\lim_{x\to 0} \frac{e^{x^2} - (1+x^2)}{(e^x-1)^4} = \lim_{x\to 0} \frac{\frac{1}{2} + \frac{x^2}{6} + \dots}{1 + \dots} \, .$$

It is easy to see that the quotient is then given by a power series whose constant term is $1/2$, and whose radius of convergence is positive.

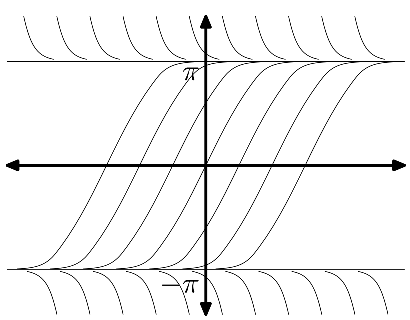
This quotient series equals the function $\dfrac{e^{x^2} - (1+x^2)}{(e^x-1)^4}$ for all nonzero x within its radius of convergence, and the value of the power series is continuous, so we get that the limit is the constant term of the quotient power series:

$$\lim_{x\to 0} \frac{e^{x^2} - (1+x^2)}{(e^x-1)^4} = \frac{1}{2} \, .$$

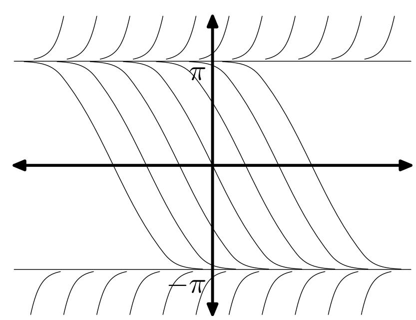
4. (16 points) Indicate which of the following graphs best depicts the solutions of the differential equation

$$y' = y \sin y$$

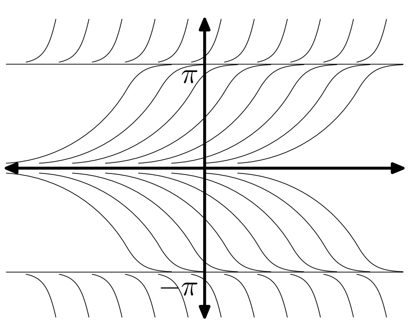
A.



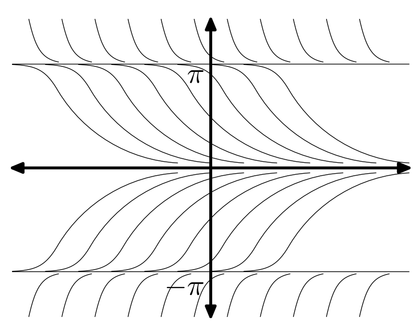
D.

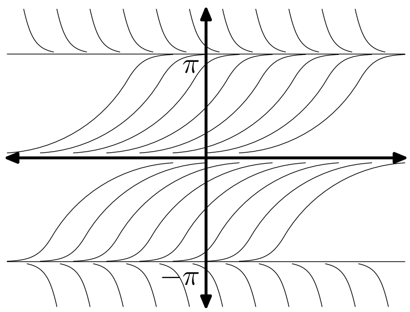


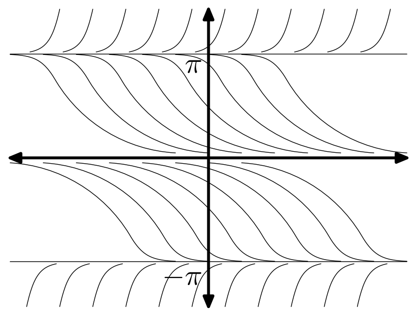
B.



E.



C. 

F. 

The right-hand side $y \sin y$ is positive for $y \in (0, \pi)$ and for $y \in (-\pi, 0)$, and negative for $y \in (\pi, 2\pi)$ and for $y \in (-2\pi, -\pi)$. It also has an equilibrium solution $y = 0$. This is consistent only with picture **C**.

(Choosing picture **A** would have gotten 3 points, since it would only have missed the equilibrium solution.)

5. (24 points) Find the general solution of the differential equation

$$y' = \frac{\ln x}{x + xy} \ .$$

This is a separable equation:

$$\frac{dy}{dx} = \frac{\ln x}{x} \cdot \frac{1}{1+y} \ .$$

Solving it as such (using the substitution $u = \ln x$, $du = dx/x$) gives

$$\int (1+y)\, dy = \int \frac{\ln x}{x}\, dx = \int u\, du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C \ ;$$

$$y + \frac{y^2}{2} = \frac{(\ln x)^2}{2} + C \ ;$$

$$\frac{y^2}{2} + y - \frac{(\ln x)^2}{2} - C = 0 \ ;$$

$$y = -1 \pm \sqrt{1 + (\ln x)^2 + 2C}$$

$$= -1 \pm \sqrt{(\ln x)^2 + C'} \ .$$