Math 1B. Solutions to First Midterm

Some Formulas

1. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$	3. $\sin A \cos B = \frac{1}{2}\big[\sin(A-B) + \sin(A+B)\big]$
2. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$	4. $\sin A \sin B = \frac{1}{2}\big[\cos(A-B) - \cos(A+B)\big]$
	5. $\cos A \cos B = \frac{1}{2}\big[\cos(A-B) + \cos(A+B)\big]$
6. $\displaystyle\int \tan u\,du = \ln\lvert\sec u\rvert + C$	7. $\displaystyle\int \sec u\,du = \ln\lvert\sec u + \tan u\rvert + C$

1. (7 points) Find

$$\int \sec^3 x \tan^3 x\,dx$$

Following the procedure for integrals of $\sec^m x \tan^n x$, set aside $\sec x \tan x$, convert all other powers of $\tan x$ to $\sec x$, and substitute $u = \sec x$, $du = \sec x \tan x\,dx$:

$$\begin{aligned}
\int \sec^3 x \tan^3 x\,dx &= \int \sec^2 x \tan^2 x \cdot \sec x \tan x\,dx \\
&= \int \sec^2 x(\sec^2 x - 1) \cdot \sec x \tan x\,dx \\
&= \int u^2(u^2 - 1)\,du \\
&= \int (u^4 - u^2)\,du \\
&= \frac{u^5}{5} - \frac{u^3}{3} + C \\
&= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C\,.
\end{aligned}$$

2. (8 points) Find

$$\int \frac{dx}{x^3 + x}$$

The integrand is a rational function, so it should be handled by partial fractions. The denominator factors as $x^3 + x = x(x^2 + 1)$, and $x^2 + 1$ is a quadratic polynomial with no real roots. Therefore, the partial fractions decomposition should have the form

$$\frac{1}{x^3 + x} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x}\,.$$

Multiplying both sides by x^3+x gives

$$1 = (Ax+B)x + C(x^2+1)\ .$$

Plugging in $x=0$, $x=1$, and $x=-1$ gives

$$\begin{aligned} 1 &= C\ ; \\ 1 &= A+B+2C\ ; \\ 1 &= A-B+2C\ . \end{aligned}$$

So $C=1$; plugging that into the other two equations gives

$$\begin{aligned} A+B &= -1 \\ A-B &= -1\ . \end{aligned}$$

Adding the two equations gives $2A=-2$, so $A=-1$, and then it is clear that $B=0$. Therefore the partial fractions decomposition is

$$\frac{1}{x^3+x} = \frac{-x}{x^2+1} + \frac{1}{x}\ ,$$

and the integral is

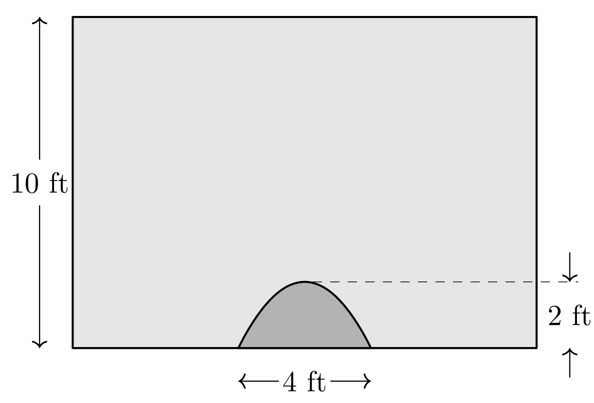
$$\begin{aligned} \int \frac{dx}{x^3+x} &= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} \\ &= \ln|x| - \frac{1}{2}\ln(x^2+1) + C\ . \end{aligned}$$

(Here we used a substitution $u=x^2+1$, $du=2x\,dx$ to find the second integral.)

3. (10 points) A diving pool is 10 feet deep and full of water. It has a viewing window on one of its vertical walls. The bottom of the window is at the bottom of the wall, and is 4 feet wide. The top of the window is a parabola of height 2 feet extending down to the bottom of the wall.

Find the hydrostatic force on the window.

Use $60\,\mathrm{lb/ft^3}$ as the density of water. You do not need to carry out the arithmetic.



First of all, we need an equation for the top of the window. Let y be the distance from the top of the window down to a given point, and let x be the (positive or negative) distance from the vertical line passing through the top of the window, with both x and y measured in feet. Then the parabola passes through the points $(0,0)$ and $(2,2)$, and is symmetric about the y-axis, so its equation is $y = x^2/2$.

However, when doing hydrostatic pressure problems, we want to use the depth (or some closely related variable) as the variable of integration, so in this case we need to express x in terms of y:

$$x = \sqrt{2y}\ .$$

The width of the window at y feet below its top is then $2\sqrt{2y}$, and the depth there is $8 + y$. Therefore, the hydrostatic force on the window is

$$\begin{aligned}
\int_0^2 60 \cdot (8+y) \cdot 2\sqrt{2y}\, dy &= 2 \cdot 60 \int_0^2 \left(8\sqrt{2y} + \sqrt{2}y^{3/2} \right) dy \\
&= 2 \cdot 60 \left[8\sqrt{2} \cdot \frac{y^{3/2}}{3/2} + \sqrt{2} \cdot \frac{y^{5/2}}{5/2} \right]_0^2 \\
&= 2 \cdot 60 \left(8\sqrt{2} \cdot \frac{2\sqrt{2}}{3/2} + \sqrt{2} \cdot \frac{4\sqrt{2}}{5/2} \right) \\
&= 2 \cdot 60 \left(\frac{8 \cdot 4 \cdot 2}{3} + \frac{8 \cdot 2}{5} \right) \\
&= 2 \cdot 60 \cdot 16 \left(\frac{4}{3} + \frac{1}{5} \right) \\
&= 2 \cdot 16\, (4 \cdot 20 + 12) \\
&= 2 \cdot 16 \cdot 92 \\
&= 2944\,\text{lb}\ .
\end{aligned}$$

It would have been OK to stop at $2 \cdot 60 \left(8\sqrt{2} \cdot \dfrac{2\sqrt{2}}{3/2} + \sqrt{2} \cdot \dfrac{4\sqrt{2}}{5/2} \right)$ lb .

4. (9 points) How many terms of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

do we need to add in order to find the sum with $|\text{error}| < 0.0004$?

(You are not being asked to repeat any proofs from the book, but do show your work and explain what you are doing.)

By the Alternating Series Estimation Theorem, we want to find n such that

$$\frac{1}{(n+1)^2} < 0.0004 = \frac{4}{10^4}\ .$$

Taking square roots of both sides and solving for n gives

$$\begin{aligned} \frac{1}{n+1} &< \frac{2}{100}\ ; \\ \frac{100}{2} &< n+1\ ; \\ 50 &< n+1\ . \end{aligned}$$

So n has to be at least 50, and we need to add at least 50 terms to get an error that small.

(One can note that after 49 terms, the 50th term will be exactly 0.0004, and then the 51st term brings down the sum, and hence the error, enough to get strictness of the error estimate after 49 terms. However, this extra work is not necessary for full credit.)

5. (8 points) Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=0}^{\infty} \frac{2 \cdot 4^n}{(2n+1)!}$$

Apply the Ratio Test:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{\frac{2\cdot 4^{n+1}}{(2n+3)!}}{\frac{2\cdot 4^n}{(2n+1)!}} \\ &= \lim_{n\to\infty} \frac{4}{(2n+2)(2n+3)} \\ &= 0\ . \end{aligned}$$

Therefore the series converges absolutely.

6. (8 points) Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}\left(\sqrt[4]{n}+1\right)}$$

First of all, both $\sqrt{n}$ and $\sqrt[4]{n}+1$ are positive increasing functions of n, so their product is also positive and increasing. Therefore $1/\left(\sqrt{n}\left(\sqrt[4]{n}+1\right)\right)$ is a decreasing sequence. Since it goes to zero as $n \to \infty$, the series converges by the Alternating Series Test.

To check for absolute convergence, we use the Limit Comparison Test with the p-series $1/n^{3/4}$:

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}\left(\sqrt[4]{n}+1\right)}}{\frac{1}{n^{3/4}}} = \lim_{n\to\infty} \frac{\sqrt[4]{n}}{\sqrt[4]{n}+1} = \lim_{n\to\infty} \frac{1}{1+1/\sqrt[4]{n}} = \frac{1}{1+0} = 1\ .$$

Since the p-series $\sum \frac{1}{n^{3/4}}$ diverges, so does the series $\sum \frac{1}{\sqrt{n}\left(\sqrt[4]{n}+1\right)}$.

Therefore, the original series converges conditionally.