7B MT 2 Solutions

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Problem 1:

a) Solve for both solutions using Gauss' law. Since the sphere is conducting and not in the presence of other fields, the charge will distribute itself evenly over the surface and produce an electric field radially outward. Inside the shell the field will be zero (we can see this by symmetry). However, if we decide to show explicitly that the field inside is zero, then the symmetry would result in a radially inward field inside the shell. Thus, the appropriate Gaussian surface is a sphere, since $\vec{E} \cdot d\vec{A} = EdA$ everywhere on the surface, thus we find

$$\oint \vec{E} \cdot d\vec{A} = 4\pi r^2 E = \frac{Q}{\epsilon_o} \Rightarrow \vec{E} = k_e \frac{Q}{r^2}\hat{r}$$

Outside the sphere. Inside, there will be no charge enclosed, thus $\vec{E} = 0$.

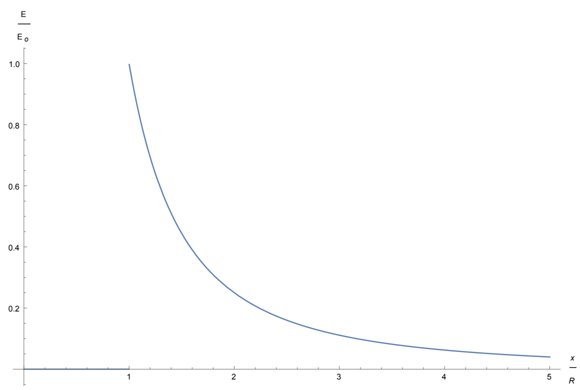


Figure 1: Plot for 1a). E_o is the electric field strength at the surface of the shell.

b) Since the field outside is that of a point charge, the potential will be a point charge as well. Inside the shell, it will maintain the same potential since $\vec{E} = 0$. We will integrate along a radial field line coming out from infinity, so that $d\vec{l} = -dr\hat{r}$, giving us

$$V = -\int_{\infty}^{r} k_e \frac{Q}{r'^2} dr' = k_e \frac{Q}{r}$$

Inside the sphere, we have

$$V = k_e \frac{Q}{R}$$

since the potential stops changing once we reach the surface of the sphere.

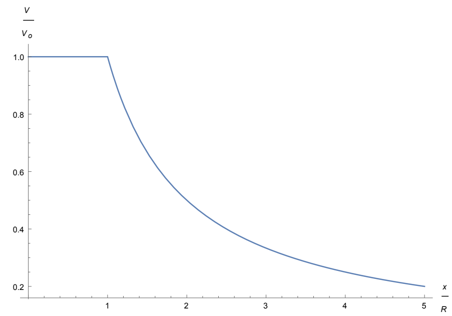


Figure 2: Plot for 1b). V_o is the electric potential at the surface of the shell.

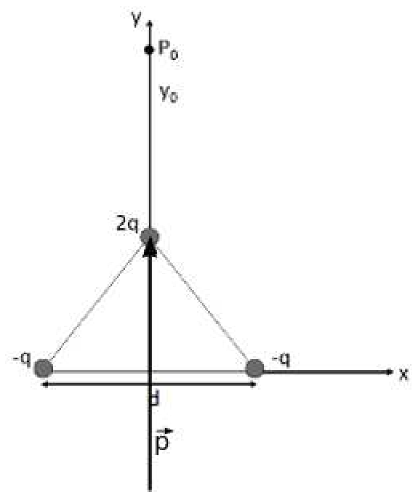
c) The minimum speed needed to escape is defined as the velocity needed for both the kinetic and potential energies of the charge to go to zero at infinity. Since the charge-shell system is not subject to any outside influence, energy will be conserved. Thus $K + U = 0$ at infinity and at the center of the sphere. Starting with the potential inside the sphere, $V = k_e Q/R$, we have

$$K + U = \frac{1}{2} m v_o^2 - k_e \frac{qQ}{R} = 0 \Rightarrow v_o = \sqrt{k_e \frac{2qQ}{mR}}$$

Problem 2:

a) By superposition of the electric field, the net dipole moment will just be the sum of the two individual dipoles (we can imagine that we really have four charges if we break up the $2q$ charge into two equal charges. Then we would have two dipoles of charge q whose positive charges are at the same point). Thus we find

$$\vec{p}_{net} = \vec{p}_1 + \vec{p}_2 = qd\left(\left(\frac{1}{2} - \frac{1}{2}\right)\hat{x} + \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right)\hat{y}\right) = \sqrt{3}qd\,\hat{y}$$



b) In order to make an approximation, we must first find the exact field. This will just be a sum over point charges.

$$\vec{E}_{left} = -k_e\frac{q}{y_o^2 + \frac{1}{4}d^2}\left(\frac{d}{2\sqrt{y_o^2 + \frac{1}{4}d^2}}\hat{x} + \frac{y_o}{\sqrt{y_o^2 + \frac{1}{4}d^2}}\hat{y}\right)$$

$$\vec{E}_{right} = -k_e\frac{q}{y_o^2 + \frac{1}{4}d^2}\left(-\frac{d}{2\sqrt{y_o^2 + \frac{1}{4}d^2}}\hat{x} + \frac{y_o}{\sqrt{y_o^2 + \frac{1}{4}d^2}}\hat{y}\right)$$

$$\vec{E}_{middle} = 2k_e\frac{q}{\left(y_o - \sqrt{\frac{3}{4}}d\right)^2}\hat{y}$$

Then the total field is

$$\vec{E} = 2k_e q\left(\frac{1}{\left(y_o - \sqrt{\frac{3}{4}}d\right)^2} - \frac{y_o}{\left(y_o^2 + \frac{1}{4}d^2\right)^{\frac{3}{2}}}\right)\hat{y} = 2k_e\frac{q}{y_o^2}\left[\left(1 - \sqrt{\frac{3}{4}}\frac{d}{y_o}\right)^{-2} - \left(1 + \frac{1}{4}\frac{d^2}{y_o^2}\right)^{\frac{3}{2}}\right]\hat{y}$$

The two terms are easily approximated to first order as

$$\vec{E} = 2k_e \frac{q}{y_o^2} \left[\left(1 - \sqrt{\frac{3}{4}} \frac{d}{y_o}\right)^{-2} - \left(1 + \frac{1}{4}\frac{d^2}{y_o^2}\right)^{\frac{3}{2}} \right] \hat{y} \approx 2k_e \frac{q}{y_o^2} \left[\left(1 + \sqrt{3}\frac{d}{y_o}\right) - \left(1 - \frac{3}{8}\frac{d^2}{y_o^2}\right) \right] \hat{y}$$

$$\approx 2k_e \frac{\sqrt{3}qd}{y_o^3} \hat{y}$$

We've dropped the second term because $d^2/y_o^2 \ll d/y_o$. The remaining field is just the electric dipole field along its axis, with dipole moment $\vec{p} = \sqrt{3}qd\,\hat{y}$. This is what we expected from part (a).

MT 2 (Prof. Lanzara)

Problem 3

A. The initial total charge on the capacitors is $Q_i = Q_1 + Q_2$. The total capacitance is just the sum of the two individual capacitances, because the capacitors are connected in parallel: $C_i = C_1 + C_2$. Using the formula $V = Q/C$ for capacitance, we find the initial voltage to be

$$V_i = \frac{Q_i}{C_i} = \frac{Q_1 + Q_2}{C_1 + C_2}. \tag{1}$$

The total charge on the capacitors after the insertion of the dielectric is the same as the total charge before, $Q_f = Q_i \equiv Q$, because there is no way for charge to move between the two isolated conductors. The capacitance of conductor 2 has changed to $C_{2f} = kC_2$, while the capacitance of conductor 1 is unchanged, $C_{1f} = C_1$. Then the final total capacitance is $C_f = C_1 + kC_2$. Using the capacitance equation again, we find

$$V_f = \frac{Q_f}{C_f} = \frac{Q_1 + Q_2}{C_1 + kC_2}. \tag{2}$$

Because the capacitors are connected in parallel, the voltage across each capacitor is the same as the total voltage V_f. Making use of the capacitance equation again, we find the charges on each capacitor after the insertion of the dielectric:

$$Q_{1f} = C_{1f} V_f = C_1 \left(\frac{Q_1 + Q_2}{C_1 + kC_2} \right) \tag{3}$$

$$Q_{2f} = C_{2f} V_f = kC_2 \left(\frac{Q_1 + Q_2}{C_1 + kC_2} \right). \tag{4}$$

The total charge remains the same, but is redistributed between the two capacitors.

B. The energy stored on a capacitor is given by $U = Q^2/2C$. Plugging in the quantities we found in part (A), we obtain the initial and final energies:

$$U_i = \frac{1}{2}\frac{Q_i^2}{C_i} = \frac{1}{2}\frac{Q^2}{C_1 + C_2} \tag{5}$$

$$U_f = \frac{1}{2}\frac{Q_f^2}{C_f} = \frac{1}{2}\frac{Q^2}{C_1 + kC_2}. \tag{6}$$

We see that the higher the dielectric constant, the more energy is lost, with all energy being lost in the limit $k \to \infty$. This makes sense - the higher the dielectric constant, the more polarizable the dielectric material. In the limit of infinite polarizability, the charges inside the dielectric can rearrange to completely cancel the electric field inside the capacitor.

Problem 4

Let's start by considering the electric field due to a slab of thickness d and uniform charge density ρ, with units of charge per volume outside of the slab. The electric field above the slab is, by superposition, the sum of the electric fields due to each slice of the slab. A slice of thickness dz (where z is the direction perpendicular to the slab and $z = 0$ is the center of the slab) has areal charge density ρdz. As we take $dz \to 0$, the slices become infinite planes, and contribute field

$$\left| d\vec{E} \right| = \frac{\rho dz}{2\epsilon_0} \hat{z} \ \mathrm{sgn}(z). \tag{7}$$

If we didn't remember this form, we could get it from Gauss's law. Take the Gaussian surface to be two identical plane figures of area A, one above the plane and one below, and the connecting face, which is perpendicular to the plane. The electric field must be perpendicular to the plane at all points, because the plane has rotational symmetry about any point, and it must not depend on the coordinates parallel to the plane, because the plane has translational symmetry along these coordinates. Moreover the field points either away from the plane on both sides or towards it, because the plane has reflection symmetry through itself. Then the electric flux through the sides of the Gaussian surface is zero, and the total flux is $2E(z)A$, where $E(z)$ is the magnitude of the electric field a distance z from the plane, because the electric field is parallel to the normal vector on both caps of area A. Gauss's law tells us that this must be equal to the enclosed charge, which is σA, divided by ϵ_0. Then the area cancels from both sides, and we find

$$\vec{E}(z) = \frac{\sigma}{2\epsilon_0}\hat{z}\ \mathrm{sgn}(z). \tag{8}$$

We can now find the total field outside the slab by integrating over the planar slices:

$$\left|\vec{E}\right| = \int_0^d \left|d\vec{E}\right| = \int_0^d \frac{\rho dz}{2\epsilon_0} = \frac{\rho d}{2\epsilon_0}. \tag{9}$$

We see that the field looks like that due to an infinite plane of charge density ρd

We turn now to the exam problem. It will be helpful to define a coordinate x, the distance from the center of the slab along the direction perpendicular to the surface, with positive x to the right. We are told that the plane and the slab both have thicknesses small on the scale of their other dimensions. This means that, for small x and close to the center of the surfaces, we can approximate them as having infinite extent. Let's consider the electric fields due to the plane and the slab separately. The field due to the plane is

$$\vec{E}_{\text{plane}}(x) = \frac{\sigma}{2\epsilon_0}\hat{x}\ \mathrm{sgn}(x + \frac{d}{2}). \tag{10}$$

At position x inside the slab, there is a slab of width $d/2 + x$ to the left and a slab of width $d/2 - x$ to the right. As we argued above, this looks like an infinite plane of areal charge density $\rho_E(d/2 + x)$ on the left and one of $\rho_E(d/2 - x)$ on the right. The electric field is then

$$\vec{E}^{(\text{in})}_{\text{slab}}(x) = \frac{\rho_E}{2\epsilon_0}\left((\frac{d}{2} + x) - (\frac{d}{2} - x\right)\hat{x} = \frac{\rho_E}{\epsilon_0}x\hat{x}, \tag{11}$$

where we've taken into account the fact that the material to the left of x exerts an electric field in the positive x direction (assuming $\rho_E > 0$) and the fact that multiplying ρ_E by a length gives a surface charge density. Outside the slab, the slab looks exactly like an infinite plane, and we have

$$\vec{E}^{(\text{out})}_{\text{slab}} = \frac{\rho_E d}{2\epsilon_0}\hat{x}\ \mathrm{sgn}(x). \tag{12}$$

Putting these pieces together, we find that the total electric field is

$$\vec{E}(x) = \frac{\hat{x}}{\epsilon_0}\begin{cases} -\frac{1}{2}(\sigma + \rho_E d) & \text{left of plane} \\ \rho_E x & \text{inside slab} \\ \frac{1}{2}(\sigma + \rho_E d) & \text{right of slab} \end{cases} \tag{13}$$

Problem 5

a)

We cannot use Gauss's law here, as the charge distribution does not have planar, radial, or cylindrical symmetry. Thus, we must use Coulomb's law. Take a bit of charge dq on the rod that sits at a position x. Then, the distance vector from this bit of charge to the point a is $(x-a)\hat{x}$, where x is the distance from the origin to the location of dq (as in part b). Thus, Coulomb's law is

$$\vec{E}_{line} = \int \frac{dq}{4\pi\epsilon_0 |\vec{r}|^2}\hat{r} = \int \frac{dq}{4\pi\epsilon_0 (x-a)^2}\hat{x}$$

Since this is a linear charge distribution, and the charge is along the x-axis, I replace dq with λdx, where $\lambda = Q/L$ since the charge is distributed uniformly across the rod. I integrate between $x = 0$ and L, as this is the region where there is charge.

$$\begin{aligned}
\vec{E}_{line} &= \int_0^L \frac{\lambda dx}{4\pi\epsilon_0 (x-a)^2}\hat{x} \\
&= \frac{Q}{4\pi\epsilon_0 L}\int_{-a}^{L-a} \frac{du}{u^2}\hat{x} \\
&= \frac{Q}{4\pi\epsilon_0 L}\left(-\frac{1}{L-a} - \frac{1}{a}\right)\hat{x} \\
&= \frac{Q}{4\pi\epsilon_0 L}\left(\frac{1}{a-L} - \frac{1}{a}\right)\hat{x}
\end{aligned}$$

In the second line I used the substitution $u = x - a$.

b)

This is just a simple application of Coulomb's law. The field will point away from the charge, so it will be in the $-\hat{x}$ direction. The distance from the point charge to $x = a$ is $2L - a$. Thus

$$\vec{E}_{point} = -\frac{Q}{4\pi\epsilon_0 (2L-a)^2}\hat{x}$$

c)

Now we would like to find when $\vec{E}_{line} + \vec{E}_{point} = 0$. This happens when

$$\begin{aligned}
\frac{Q}{4\pi\epsilon_0 L}\left(\frac{1}{a-L} - \frac{1}{a}\right)\hat{x} &= \frac{Q}{4\pi\epsilon_0 (2L-a)^2}\hat{x} \\
\frac{1}{L(a-L)} - \frac{1}{aL} &= \frac{1}{(2L-a)^2} \\
\frac{aL^2(a-L)}{aL - L(a-L)} &= (2L-a)^2 \\
a(a-L) &= 4L^2 + a^2 - 4La \\
3a &= 4L \\
a &= \frac{4L}{3}
\end{aligned}$$