EE 20N: Structure and Interpretation of Signals and Systems
Department of Electrical Engineering and Computer Sciences — MIDTERM 2
UC BERKELEY — 27 October 2014

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Lab Day & Time: midnight SID (All Digits): 11111115

- **(10 Points)** Print your *official* name (not your e-mail address) and *all* digits of your student ID number legibly, and indicate your lab time, on *every* page.

- This exam should take up to 80 minutes to complete. You will be given at least 80 minutes, up to a maximum of 90 minutes, to work on the exam.

- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except ~~one~~ Two double-sided 8.5" × 11" sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.

- **The exam printout consists of pages numbered 1 through 8.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the eight numbered pages. If you find a defect in your copy, notify the staff immediately.

- Please write neatly and legibly, because *if we can't read it, we can't grade it.*

- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*

- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.

- We hope you do a *fantastic* job on this exam.

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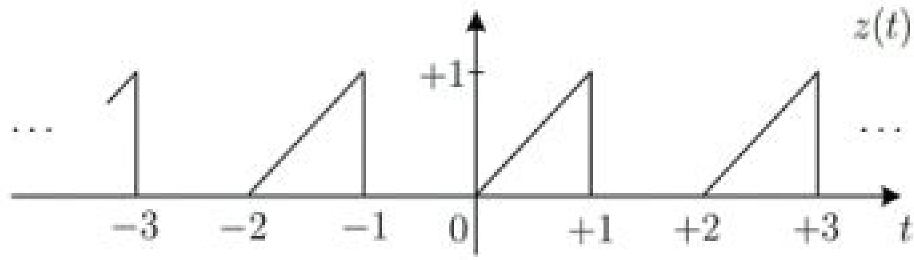
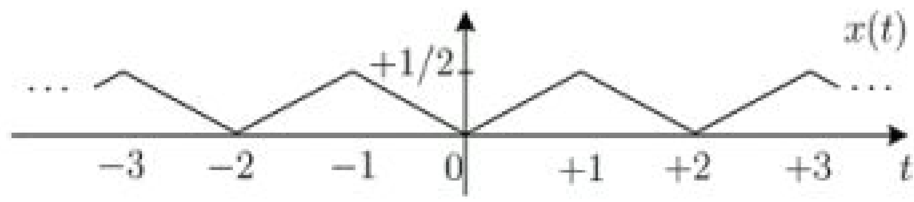
MT2.1 (54 Points) For this problem consider the the three continuous-time waveforms z, x, and y described below:



The trigonometric Fourier series expansion for this waveform is

$$z(t) \;=\; A_0 + \sum_{k=1}^{+\infty} A_k \cos\left(k\omega_z t\right) + \sum_{\ell=1}^{+\infty} B_\ell \sin\left(\ell\omega_z t\right) \;, \tag{1}$$

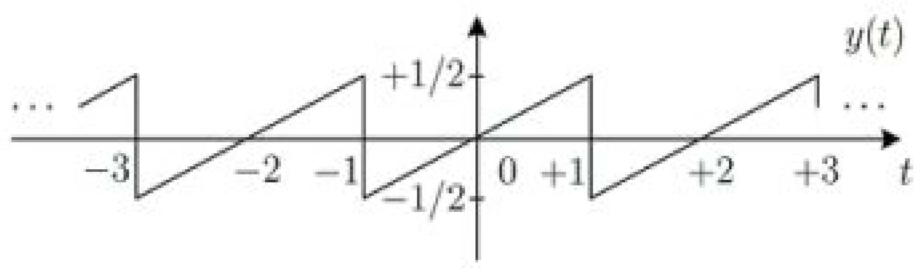
where p_z is the fundamental period, and $\omega_z = 2\pi/p_z$ the fundamental frequency.



The trigonometric Fourier series expansion for x is

$$x(t) \;=\; C_0 + \sum_{k=1}^{+\infty} C_k \cos\left(k\omega_x t\right) + \sum_{\ell=1}^{+\infty} D_\ell \sin\left(\ell\omega_x t\right) \;, \tag{2}$$

where p_x is the fundamental period, and $\omega_x = 2\pi/p_x$ the fundamental frequency.



The trigonometric Fourier series expansion for y is

$$y(t) \;=\; E_0 + \sum_{k=1}^{+\infty} E_k \cos\left(k\omega_y t\right) + \sum_{\ell=1}^{+\infty} F_\ell \sin\left(\ell\omega_y t\right) \;, \tag{3}$$

where p_y is the fundamental period, and $\omega_y = 2\pi/p_y$ the fundamental frequency.

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MT2.1 (Continued): Determine all the Fourier series coefficients A_k, B_ℓ, C_k, D_ℓ, E_k, and F_ℓ in Equations 1–3, where $k \in \{0, 1, 2, \ldots\}$ and $\ell \in \{1, 2, 3, \ldots\}$. Be careful to treat A_0, C_0, and E_0 separately from the rest of the coefficients A_k, C_k, and E_k, respectively.

(Point Allocations) Each of A_0, C_0, and E_0 has an allocation of 4 points maximum, whereas each of the other coefficients carries a maximum of 7 points.

You may use the blank space below, and on the next page, to show your work. We leave it to you to decide the order in which to determine the coefficients. But we do advise that you exploit structure in the signals. You may cite *general* results from your problem sets without having to prove them here again.

Method I: Determine the coefficients for x and y. Recognize that $z = x + y$. The write the coefficients of z by inspection.

Note that x, y, z have fundamental period $p = 2$ so their fundamental frequency is $\omega_0 = \frac{2\pi}{2} = \pi$

$$x(t) = C_0 + \sum_{k=1}^{\infty} C_k \cos(k\pi t) + \sum_{\ell=1}^{\infty} D_\ell \sin(\ell \pi t)$$

x is an even function, so all the sine coefficients D_ℓ are zero: $D_\ell = 0 \quad \ell = 1, 2, 3, \ldots$

$$C_0 = \frac{1}{2}\int_{-1}^{1} x(t)\,dt = \int_0^1 \frac{t}{2}\,dt = \frac{t^2}{4}\Big|_0^1 = 1/4$$

$$C_k = \frac{2}{2}\int_{-1}^{1} x(t)\cos(k\pi t)\,dt = \cancel{2}\int_0^1 \frac{t}{\cancel{2}}\cos(k\pi t)\,dt$$

$u = t \rightarrow du = dt \qquad dv = \cos(k\pi t)\,dt \rightarrow v = \frac{\sin(k\pi t)}{k\pi}$

$$C_k = \cancel{\frac{t\sin(k\pi t)}{k\pi}\Big|_0^1} - \frac{1}{k\pi}\int_0^1 \sin(k\pi t)\,dt = \frac{\cos(k\pi t)}{(k\pi)^2}\Big|_0^1$$

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MT2.1 (Continued):

$$C_k = \frac{(-1)^k - 1}{(k\pi)^2} = \begin{cases} \frac{-2}{(k\pi)^2} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

$$y(t) = E_0 + \sum_{k=1}^{\infty} E_k \cos(k\pi t) + \sum_{\ell=1}^{\infty} F_\ell \sin(\ell\pi t)$$

y is an odd function, so $E_k = 0$ for all $k \in \{0, 1, 2, \ldots\}$

$$F_\ell = \frac{\cancel{2}}{\cancel{2}} \int_{-1}^{1} y(t) \sin(\ell\pi t)\, dt = \cancel{2} \int_0^1 \frac{t}{\cancel{2}} \sin(\ell\pi t)\, dt$$

$u = t \rightarrow du = dt$, $dv = \sin(\ell\pi t)\,dt \rightarrow v = \frac{-\cos(\ell\pi t)}{\ell\pi}$

$$F_\ell = \frac{-t\cos(\ell\pi t)}{\ell\pi}\Bigg|_0^1 + \underbrace{\cancel{\int_0^1 \frac{\cos(\ell\pi t)}{\ell\pi} dt}}_{0} = \frac{-(-1)^\ell}{\ell\pi} = \frac{(-1)^\ell}{\ell\pi}$$

Since $z(t) = x(t) + y(t)$, then

$$A_0 = C_0 + \underbrace{\cancel{E_0}}_{0} = 1/4, \quad A_k = C_k + \underbrace{\cancel{E_k}}_{0} = C_k = \frac{(-1)^k - 1}{(k\pi)^2}$$

$$B_\ell = \underbrace{\cancel{D_\ell}}_{0} + F_\ell = F_\ell = \frac{(-1)^\ell}{\ell\pi}$$

Method II: Determine the coefficients A_k and B_ℓ for z first. Recognize that $z(t) = x(t) + y(t)$, where x is the even part of z, and y is the odd part. (Continued on p. 8)

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MT2.2 (35 Points) The impulse response of a discrete-time LTI system H is described by

$$\forall n \in \mathbb{Z}, \qquad h(n) = -\alpha^n u(-n-1), \tag{4}$$

where α is a complex-valued parameter.

(a) (5 Points) Select the strongest true assertion from the list below.

(i) The system must be causal.

(ii) The system could be causal, but does not have to be.

(iii) The system cannot be causal.

Provide a succinct, yet clear and convincing explanation.

h(n) is nonzero for atleast one value of $n < 0$.

This system is, in fact, anticausal. $h(n) = 0 \quad n > 0$.

(b) (10 Points) Determine the values of α for which the system is BIBO stable. You must consider the entire complex plane, not merely the real numbers.

Then select two *real* values of α—one that makes the system BIBO stable, and another that makes it unstable—and draw a well-labeled plot of $h(n)$ for each case.

For BIBO stability we must have $\sum |h(n)| < \infty$.

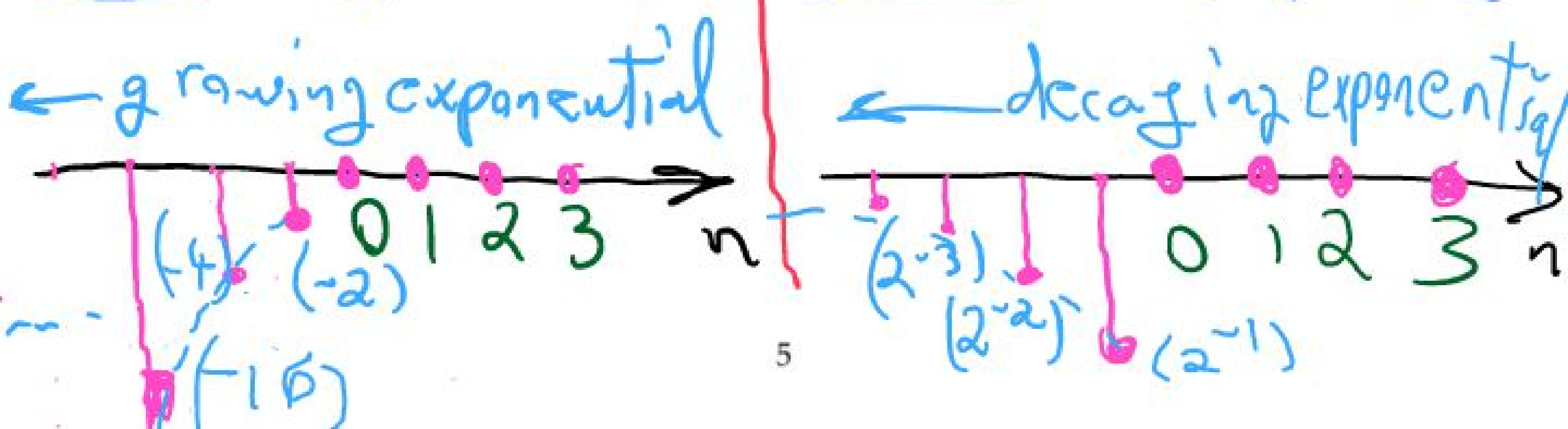
Applying this, we have $\sum_{n=-\infty}^{-1} |\alpha|^n = \sum_{\ell=1}^{\infty} (|\alpha|^{-1})^n = \sum_{\ell=0}^{\infty} (|\alpha|^{-1})^n - 1$

let $\ell = -n$

Clearly, the geometric sum is finite iff $|\alpha|^{-1} < 1 \iff |\alpha| > 1$

So for BIBO stability, α must be strictly outside the unit circle.

Case I: $\alpha = 1/2$ (unstable) | Case II: $\alpha = 2$ (stable



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(c) (10 Points) Suppose the system is BIBO stable. Determine (in terms of α) a reasonably simple expression for $H(\omega)$, the system's frequency response.

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n} = -\sum_{n=-\infty}^{-1} \alpha^n e^{-i\omega n} = -\sum_{\ell=1}^{\infty} \alpha^{-\ell} e^{i\omega \ell}$$

let $\ell = -n$

$$= -\sum_{\ell=1}^{\infty} (\alpha^{-1} e^{i\omega})^{\ell} = -\left[\sum_{\ell=0}^{\infty} (\alpha^{-1} e^{i\omega})^{\ell} - 1\right]$$

since H is BIBO stable

$$= -\left[\frac{1}{1-\alpha^{-1} e^{i\omega}} - 1\right] = 1 - \frac{1}{1-\alpha^{-1}e^{i\omega}} = \frac{\cancel{1} - \alpha^{-1} e^{i\omega} \cancel{-1}}{1-\alpha^{-1} e^{i\omega}}$$

$$H(\omega) = \frac{1}{1-\alpha e^{-i\omega}}$$

(d) (10 Points) Does either of the following linear, constant-coefficient difference equations characterize the input-output behavior of the system H correctly? If so, show how to solve the appropriate difference equation to obtain the impulse response h given by Equation 4 at the beginning of this problem. If you believe neither equation is correct, explain your reasoning succinctly, but clearly and convincingly.

(I) $\begin{cases} y(n) = \alpha\, y(n-1) + x(n) \\ y(n) = 0 \quad \text{for } n < 0. \end{cases}$ (II) $\begin{cases} y(n) = \frac{1}{\alpha}[y(n+1) - x(n+1)] \\ y(n) = 0 \quad \text{for } n \geq 0. \end{cases}$

Eqn (I) cannot possibly be the correct choice, because our impulse response is nonzero for $n \leq -1$, whereas Eqn (I) insists the any response be zero for $n < 1$.

Eqn (II) is the correct choice. Let $x(n) = \delta(n)$, in which case $y(n) = h(n)$, and solve the equation backward in time. In particular,

$$h(-1) = \alpha^{-1}[\underbrace{h(0)}_{0} - \underbrace{\delta(0)}_{1}] = -\alpha^{-1}$$

$$h(-2) = \alpha^{-1} h(-1) = -\alpha^{-2}$$

$\vdots$

$$h(-k) = -\alpha^{-k} \quad k = 1, 2, 3, \ldots$$

Let $n = -k$

$$h(n) = -\alpha^{n} \quad n = -1, -2, -3, \ldots$$

So $h(n) = -\alpha^n u(-n-1)$

Note $u(-n-1) = \begin{cases} 0 & n = 0, 1, 2, \ldots \\ 1 & n = -1, -2, \ldots \end{cases}$

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MT2.3 (16 Points) A discrete-time periodic signal x has *fundamental* period p_x. We pass the signal x through an N-*fold Subsampler* (also known as an N-*fold Downsampler*) to produce the signal y as follows:

$$\forall n \in \mathbb{Z}, \qquad y(n) = x(Nn),$$

where N is an integer greater than 1.

Determine a reasonably simple expression for p_y, the *fundamental* period of the signal y. Do *not* try to prove that y is periodic. It is! ☺

Does the fundamental period p_y depend on whether p_x and N are coprime? If so, how? If not, show that it doesn't.

We say that two positive integers a and b are *coprime* if $\gcd(a, b) = 1$, which means that the ratio a/b cannot be simplified.

What is the smallest value that fundamental frequency $\omega_y = 2\pi/p_y$ can take?

Be succinct, but clear and convincing. You may or may not find it helpful to use one or more number-theoretic notions, such as $\mathrm{lcm}(a, b)$ and $\gcd(a, b)$, which denote, respectively, the *least common multiple* and the *greatest common divisor* of a and b.

$y(n) = x(nN)$. The fundamental period p_y is such that $y(n+p_y) = y(n)$; that is, $x((n+p_y)N) = x(nN + p_y N) = x(nN)$. This means $p_y N = \ell p_x$ for an appropriately chosen positive integer ℓ.

$p_y = \frac{\ell p_x}{N}$: To determine p_y, pick the smallest ℓ that makes the right-hand side an integer.

Case I: p_x & N are coprime. In this case $\ell = N$ which means $p_y = p_x$.

Case II: p_x & N are not coprime. In this case, $p_x = gM$ and $p_y = gL$, where $g = \gcd(p_x, N)$ and M & L are coprime.

So $p_y = \frac{\ell \cancel{g} M}{\cancel{g} L} = \frac{\ell M}{L} \Rightarrow \ell = L \Rightarrow p_y = M = \frac{p_x}{\gcd(p_x, N)}$

$\max p_y = p_x \Rightarrow \min \omega_y = \omega_x$

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Potentially Useful or Useless Facts and Formulas:

Continuous-Time Trigonometric Fourier Series A p-periodic, real-valued continuous-time signal x can be represented by the trigonometric Fourier series expansion

$$x(t) = A_0 + \sum_{k=1}^{+\infty} A_k \cos(k\omega_0 t) + \sum_{\ell=1}^{+\infty} B_\ell \sin(\ell\omega_0 t),$$

where $\omega_0 = 2\pi/p$ is the fundamental frequency, and p the fundamental period, of x. Equation 5 is the *synthesis equation* of the trigonometric Fourier series expansion.

The coefficients are given by the following *analysis equations*:

$$\begin{aligned} A_0 &= \frac{1}{p}\int_{\langle p\rangle} x(t)dt. \\ A_k &= \frac{2}{p}\int_{\langle p\rangle} x(t)\cos(k\omega_0 t)\,dt, \quad 1 \leq k. \\ B_\ell &= \frac{2}{p}\int_{\langle p\rangle} x(t)\sin(\ell\omega_0 t)\,dt, \quad 1 \leq \ell. \end{aligned}$$

Here $\langle p\rangle$ denotes a contiguous interval of duration p units along the real axis.

Closed Form for a Geometric Sum If a potentially complex parameter α is inside the unit circle (i.e., $|\alpha| < 1$), then

$$\sum_{k=0}^{+\infty} \alpha^k = \frac{1}{1-\alpha}.$$

MT2.1 (Cont.): Method II (Cont.)

$x(t) = \frac{z(t)+z(-t)}{2}$ even part of z, $y(t) = \frac{z(t)-z(-t)}{2}$ odd part of z

Easy to show that if $z(t)$ has coeffs A_k, B_ℓ, then $z(-t)$ has coeffs $A_k, -B_\ell$. Why? So, $C_k = A_k$, $D_\ell = 0$, $E_k = 0$, and $F_\ell = B_\ell$.

As for A_k, B_ℓ, we have: $A_0 = \frac{1}{2}\int^2 z(t)\,dt = \frac{1}{2}\int_0^1 t\,dt = 1/4$

$A_k = \frac{2}{2}\int_0^2 x(t)\cos(k\pi t)dt = \int_0^1 t\cos(k\pi t)dt = \frac{(-1)^k - 1}{(k\pi)^2}$

$B_\ell = \frac{2}{2}\int_0^2 x(t)\sin(\ell\pi t)dt = \int_0^1 t\sin(\ell\pi t)dt = \frac{(-1)^\ell}{\ell\pi}$