Problem 1

a

Let us consider one plate one of the plates. Draw a Gaussian cylinder around it such that the z axis of the cylinder points perpendicular to the plate. The electric field must point in the z direction. This is because if at any point, the infinite plane looks the same in the x and y direction, so the forces from these directions would have to cancel. Thus, the endcaps of the cylinder have an area vector parallel to the electric field, but the surface of the cylinder has an area vector perpendicular to the electric field. The electric field is constant on the endcap, as as argued before, at any given z, there are no special points as everything looks the same in the x and y direction. Using Gauss's law gives

$$\oint \vec{E} \cdot d\vec{A} = 2|\vec{E}|A = \frac{\sigma A}{\epsilon_0}$$

which gives

$$|\vec{E}| = \frac{\sigma}{2\epsilon_0}$$

The direction points away from the plates. Now consider the problem at hand. In the first quadrant $(x > 0, y > 0)$, the total electric field is:

$$\frac{\sigma}{2\epsilon_0}(\hat{x} - \hat{y})$$

In the second quadrant $(x < 0, y > 0)$,

$$\frac{\sigma}{2\epsilon_0}(-\hat{x} - \hat{y})$$

In the third quadrant $(x < 0, y < 0)$,

$$\frac{\sigma}{2\epsilon_0}(-\hat{x} + \hat{y})$$

In the fourth quadrant $(x > 0, y < 0)$,

$$\frac{\sigma}{2\epsilon_0}(\hat{x} + \hat{y})$$

0.1 b

Since the string is taught, the tension of the string balances the other forces. The sum of forces in the $\hat{y}$ direction are:

$$T\sin(\theta) + mg = \frac{-q\sigma}{2\epsilon_0}$$

The sum of forces in the $\hat{x}$ direction are:

$$\frac{-q\sigma}{2\epsilon_0} = T\cos(\theta)$$

Dividing the two equations

$$\tan(\theta) = \frac{\frac{-q\sigma}{2\epsilon_0} - mg}{\frac{-q\sigma}{2\epsilon_0}} = 1 + \frac{2mg\epsilon_0}{q\sigma}.$$

So the angle is

$$\theta = \arctan(1 + \frac{2mg\epsilon_0}{q\sigma}).$$

Problem 2

a

I will use Coulomb's law to determine the field. The distance vector, $\hat{r}$ from where a charge sitting at some θ to the origin is $\hat{r} = (-\cos(\theta), -\sin(\theta))$. Thus:

$$\vec{E} = \int \frac{dq}{4\pi\epsilon_0 r^2}\hat{r} = \int_0^{2\pi} \frac{\lambda(\theta)(-\cos(\theta)\hat{x} - \sin(\theta)\hat{y})ad\theta}{4\pi\epsilon_0 a^2} = -\frac{\lambda_0}{4\pi\epsilon_0 a}\int_0^{2\pi} \cos(k\theta)(\cos(\theta)\hat{x} + \sin(\theta)\hat{y})$$

Now look at the integral in the $\hat{x}$ direction. Using the given identity

$$\begin{aligned}
\int_0^{2\pi} \cos(k\theta)\cos(\theta)d\theta &= \frac{1}{2}\int_0^{2\pi} \cos((k+1)\theta) + \cos((k-1)\theta)d\theta \\
&= \frac{1}{2}\left(\frac{\sin((k+1)\theta)}{k+1} + \frac{\sin((k-1)\theta)}{k-1}\right)\Bigg|_0^{2\pi} \\
&= \frac{1}{2}\left(\frac{\sin((k+1)2\pi)}{k+1} + \frac{\sin((k-1)2\pi)}{k-1}\right)
\end{aligned}$$

Now look at the integral in the $\hat{y}$ direction. Using the given identity

$$\begin{aligned}
\int_0^{2\pi} \sin(\theta)\cos(k\theta)d\theta &= \frac{1}{2}\int_0^{2\pi} \sin((k+1)\theta) + \sin((1-k)\theta)d\theta \\
&= -\frac{1}{2}\left(\frac{\cos((k+1)\theta)}{k+1} + \frac{\cos((1-k)\theta)}{1-k}\right)\Bigg|_0^{2\pi} \\
&= \frac{1}{2(k+1)} + \frac{1}{2(1-k)} - \frac{1}{2}\left(\frac{\cos((k+1)2\pi)}{k+1} + \frac{\cos((k-1)2\pi)}{1-k}\right)
\end{aligned}$$

Putting this together,

$$\begin{aligned}
\vec{E} &= -\frac{\lambda_0}{4\pi\epsilon_0 a}\left[\frac{1}{2}\left(\left(\frac{\sin((k+1)2\pi)}{k+1} + \frac{\sin((k-1)2\pi)}{k-1}\right)\right)\hat{x}\right. \\
&\quad \left. + \left(\frac{1}{2(k+1)} + \frac{1}{2(1-k)} - \frac{1}{2}\left(\frac{\cos((k+1)2\pi)}{k+1} + \frac{\cos((k-1)2\pi)}{1-k}\right)\right)\hat{y}\right]
\end{aligned}$$

Except when $k = 1$ (or equivalently $k = -1$, as the charge distribution is an even function). In this case, the second term in the first line of the $\hat{x}$ integral is 1. Thus

$$\int_0^{2\pi} \cos^2(\theta)d\theta = \frac{1}{2}\frac{\sin(4\pi)}{2} + \pi = \pi$$

When $k = 1$, the second term in the first line of the $\hat{y}$ integral is 0. Thus,

$$\int_0^{2\pi} \cos(\theta)\sin(\theta)d\theta = \int_1^1 udu = 0$$

Where I used the substitution $u = \sin(\theta)$. Thus

$$\vec{E}_{k=\pm 1} = -\frac{\lambda}{4\epsilon_0 a}\hat{x}$$

b

By looking at the expressions, the electric field in the x direction is 0 when $k+1$ and $k-1$ are half integers or full integers, as this will make the argument of sin a multiple of π. However, as shown above, $k=1$ and $k=-1$, is an exception to the rule. This means that $k \in \{..., -2, -\frac{3}{2}, -\frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2}, 2, ...\}$.
By looking at the expressions, the electric field in the y direction is 0 when $k+1$ and $k-1$ are full integers, as this will make the argument of sin a multiple of 2π. This means that $k \in \{..., -2, -1, 0, 1, 2, ...\}$.
You could have done this without having gotten an answer for part a. For there to be no field in the y direction, there has to be as much charge above the axis as below. Thus, the charge distribution should be symmetric about $\theta = \pi$. This happens when k is an integer, or $k \in \{..., -2, -1, 0, 1, 2, ...\}$. For there to be no field in the x direction, there has to be as much charge on the left of the y-axis as to the right of it. Thus, the charge distribution should look the same from $[\pi/2, 3\pi/2]$ and from $[0, \pi/2] \bigcup [3\pi/2, 2\pi]$. By looking at the integral over the regions, you see that this occurs when k is a half integer except 1 or -1. Thus, $k \in \{..., -2, -\frac{3}{2}, -\frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2}, 2, ...\}$.

Problem 3

a)

The region in between the spheres has no charge, but this is equivalent to having positive charge and negative charge there that overlap. Thus, the system is equivalent to a positive sphere with charge density ρ centered at $x = -a$ and a negative sphere with charge density $-\rho$ centered at $x = a$.
Consider one of the spheres. Use Gauss's law with a spherical Gaussian surface of radius $r < a$. The charge enclosed is $Q_{in} = \frac{4\pi r^3}{3}\rho$. The field should be radial as there is no variation in the θ or ϕ direction of the charge distribution. At some r, any given point looks the same, so the field should be constant along r. Thus

$$\oint \vec{E}_{in} \cdot d\vec{A} = |\vec{E}_{in}| 4\pi r^2 = \frac{q_{in}}{\epsilon_0} = \frac{4\pi r^3}{3\epsilon_0}\rho$$

And so the magnitude of the electric field is

$$|\vec{E}_{in}| = \frac{\rho r}{3\epsilon_0}$$

By the superposition principle, the electric field due to both of the spheres is the sum of the two. Thus, for a point x inside the overlap region:

$$|\vec{E}_{overlap}(x)| = \frac{\rho(x - \frac{a}{2})}{3\epsilon_0} + \frac{-\rho(x + \frac{a}{2})}{3\epsilon_0} = \frac{\rho a}{3\epsilon_0}$$

As the right hand side is negatively charged, positive charges will be attracted to it. Thus:

$$\vec{E}_{overlap}(x) = \frac{\rho a}{3\epsilon_0}\hat{x}$$

b)

Now take a Guassian surface of radius $r > a$. Now, the charge enclosed is simple, $Q_{in} = \rho\frac{4\pi a^3}{3}$. For the same reasons above, the field should be radial and constant at lines of constant r. Thus

$$\oint \vec{E}_{out} \cdot d\vec{A} = |\vec{E}_{out}| 4\pi r^2 = \frac{q_{in}}{\epsilon_0} = \frac{\rho 4\pi a^3}{3\epsilon_0}$$

And so the magnitude of the electric field is

$$|\vec{E}_{out}| = \frac{\rho a^3}{3\epsilon_0 r^2}$$

By the superposition principle, the electric field due to both of the spheres is the sum of the two. Thus, for a point x inside the overlap region:

$$\begin{aligned}
|\vec{E}_{outside}(x)| &= \frac{\rho a^3}{3\epsilon_0(x + \frac{a}{2})^2} - \frac{\rho a^3}{3\epsilon_0(x -]\frac{a}{2})^2} \\
&= \frac{\rho a^3}{3\epsilon_0 x^2}\left((1 + \frac{a}{2x})^{-2} - (1 - \frac{a}{2x})^{-2}\right) \\
&\approx \frac{\rho a^3}{3\epsilon_0 x^2}\left(1 + \frac{-a}{x} - (1 + \frac{a}{x})\right) \\
&= \frac{-2\rho a^4}{3\epsilon_0 x^3}
\end{aligned}$$

Since the negative charge will be a little closer than the positive charge if x is large, the net force the charge will feel will be toward the left. Thus

$$\vec{E}_{outside} = \frac{-2\rho a^4}{3\epsilon_0 x^3}\hat{x}$$

Problem 4.

(a). Let $V(\infty)=0$, Since Gauss's law is not useful, we will have to use Coulomb's law for potentials.

$$V = k\int\frac{dq}{\rho} \quad = k\int\frac{\sigma ds r d\theta}{\rho} \quad = k(\int_0^{2\pi} d\theta)(\int_a^{\infty}\frac{\sigma}{\sqrt{r^2+x^2}}((\tfrac{dr}{dx})^2+1)^{\frac{1}{2}} r dx) \quad = 2k\pi\int_a^{\infty}\frac{\sigma}{\sqrt{r^2+x^2}}((\tfrac{dr}{dx})^2+1)^{\frac{1}{2}} r dx$$

Given $r=\frac{a^2}{x}$ and $\frac{dr}{dx}=-\frac{a^2}{x^2}$, we have

$$V = 2k\pi\int_a^{\infty}\frac{\sigma}{\sqrt{\frac{a^4+x^4}{x^2}}}(\tfrac{a^4+x^4}{x^4})^{\frac{1}{2}}\frac{a^2}{x}dx$$

$$= 2k\pi\int_a^{\infty}\sigma\frac{a^2}{x^2}dx$$

$$= 2k\pi\sigma a^2(-\tfrac{1}{x})\Big|_a^{\infty}$$

$$= 2k\pi\sigma a$$

(b).

Energy is conserved in the process.

$$E_{k1}+U_1 = E_{k2}+U_2$$

$$\tfrac{1}{2}mv_0^2 + V(x=\infty) = \tfrac{1}{2}m(0)^2 + qV(x=0)$$

$$v_0 = \sqrt{-\frac{2qV(x=0)}{m}} \quad = \sqrt{-\frac{4qk\pi\sigma a}{m}}$$

Problem 5

As C_3 and C_4 are disconnected from the voltage source, they will not gain any charge before switch S_1 is closed. C_1 and C_2 are connected in parallel to the voltage source, so they will both be charged up such that the voltage drop across them is V. Thus, before switch S_2 is closed, $Q_1 = C_1 V = 3$ C and $Q_2 = C_2 V = 6$ C (these are some massive capacitors!). Note that as the positive end of the battery is connected to the top plate of C_1 and C_2 the top plates will have positive charge and the bottom plates will have negative charge. When switch S_1 is opened and S_2 closed, there is nothing to supply new charge to the system. Thus, by conservation of charge I get that

$$\begin{aligned} Q_1 + Q_2 + Q_3 &= 9 \text{ C} \\ -Q_1 - Q_2 - Q_4 &= -9 \text{ C} \\ -Q_3 + Q_4 &= 0 \end{aligned}$$

The last equation comes from the fact that the bottom plate of C_3 and the top plate of C_4 are isolated from the rest of the circuit, and thus there is no way for excess charge to travel to this region. It turns out this last equation is redudundant as adding the first two gives you this, but this tells you that $Q_3 = Q_4$. You may have also remembered the fact that two capacitors in series should have the same charge, and you could have arrived at this statement this way.
As the capacitors are still in parallel, the voltage drop across each of the legs must be equal. This gives

$$\frac{Q_1}{C_1} = \frac{Q_2}{C_2} = \frac{Q_3}{C_3} + \frac{Q_4}{C_4} = Q_3 \left(\frac{1}{C_3} + \frac{1}{C_4} \right)$$

I will use these relations with the first equation of charge conservation to determine the value of Q_1.

$$\begin{aligned} Q_1 + \frac{C_2}{C_1} Q_1 + \frac{1}{C_1} \left(\frac{1}{C_3} + \frac{1}{C_4} \right)^{-1} Q_1 &= 9 \text{ C} \\ Q_1 + 2Q_1 + \frac{2}{3} Q_1 &= 9 \text{ C} \\ Q_1 &= \frac{27}{11} \text{ C} \end{aligned}$$

Now the rest of the charges are easy to find

$$\begin{aligned} Q_2 &= \frac{C_2}{C_1} Q_1 = \frac{54}{11} \text{ C} \\ Q_3 = Q_4 &= 9 \text{ C} - Q_1 - Q_2 = \frac{18}{11} \text{ C} \end{aligned}$$