Midterm 2 Solutions

8B Lecture 2 Spring 2014

1. To find the force on the wires use the equation for the magnetic force on a current I along a straight path given by the vector $\vec{l}$ due to some magnetic field $\vec{B}$

$$\vec{F} = I\vec{l} \times \vec{B}$$

Here the magnetic field is constant $\vec{B} = 2.0\hat{k}$ T for a rectangular region and zero outside, and each wire carries the same current $I = 2.5$ A.

(a) There are three straight segments of wire in the region where the field is nonzero. The direction vectors for these segments are as follows.

$$\vec{l}_1 = 3.0\hat{\jmath} \text{ m}$$
$$\vec{l}_2 = 4.0\hat{\imath} \text{ m}$$
$$\vec{l}_3 = -3.0\hat{\jmath} \text{ m}$$

Note that the force on segments 1 and 3 will be the same magnitude by the force law, and in opposite directions by the right hand rule, so these forces will add to zero. Therefore the total force on the wire is the force on segment 2.

$$\vec{F} = (2.5 \text{ A})(4.0\hat{\imath} \text{ m}) \times (2.0\hat{k} \text{ T}) = 20\hat{\imath} \times \hat{k} \text{ N} = \boxed{-20\hat{\jmath} \text{ N}}$$

Where the final cross product was done by the right hand rule.

(b) Here there are two straight segments given by the following direction vectors.

$$\vec{l}_1 = (4\hat{\imath} + 3.0\hat{\jmath}) \text{ m}$$
$$\vec{l}_2 = -3.0\hat{\jmath} \text{ m}$$

Since these forces will not be in the same direction by the right hand rule, calculate them explicitly.

$$\vec{F}_1 = (2.5 \text{ A})((4.0\hat{\imath}+3.0\hat{\jmath}) \text{ m}) \times (2.0\hat{k} \text{ T}) = (20\hat{\imath} \times \hat{k} + 15\hat{\jmath} \times \hat{k}) \text{ N} = (15\hat{\imath} - 20\hat{\jmath}) \text{ N}$$
$$\vec{F}_2 = (2.5 \text{ A})(-3.0\hat{\jmath} \text{ m}) \times (2.0\hat{k} \text{ T}) = -15(\hat{\jmath} \times \hat{k}) \text{ N} = -15\hat{\imath} \text{ N}$$

Where the linearity of the cross product ($(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$) was used. So then the total force on the wire is the sum of the forces on these two segments

$$\vec{F} = \vec{F}_1 + \vec{F}_2 = \boxed{-20\hat{\jmath} \text{ N}}$$

(c) Like the previous two parts, calculate the forces on each of the three straight segments and sum to get the final force, but notice the following.

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = I\vec{l}_1 \times \vec{B} + I\vec{l}_2 \times \vec{B} + I\vec{l}_3 \times \vec{B} = I(\vec{l}_1 + \vec{l}_2 + \vec{l}_3) \times \vec{B}$$

Then from the geometry $\vec{l}_1 + \vec{l}_2 + \vec{l}_3 = 4.0\hat{\imath}$ m so solve for the total force.

$$\vec{F} = (2.5 \text{ A})(4.0\hat{\imath} \text{ m}) \times (2.0\hat{k} \text{ T}) = 20\hat{\imath} \times \hat{k} \text{ N} = \boxed{-20\hat{\jmath} \text{ N}}$$

2. (a) First use Faraday's law to find the EMF induced in this loop. Assuming the rails start at the center at time $t = 0$ and then move outwards at a constant velocity, the area of the loop $A(t)$ is given by:

$$A(t) = l \times 2vt = 2lvt$$

Then, since the magnetic field $\vec{B}$ is constant and perpendicular to the area, the flux $\Phi(t)$ is simply the product of the magnitude of the field and the area.

$$\Phi(t) = BA(t) = 2Blvt$$

Faraday's law states that the EMF in the loop is related to the time derivative of the flux.

$$\varepsilon = -\frac{d\Phi}{dt} = -2Blv$$

The magnitude of this EMF is the same as the magnitude of the potential V in this circuit which consists of two resistances R in series. Therefore the loop equation for this circuit gives the relationship to the magnitude of the current in the circuit.

$$V = 2RI = 2Blv$$

$$\Rightarrow I = \boxed{\frac{Blv}{R}}$$

The direction of the current can then be found by the right hand rule and Lenz's law. Since the area of the loop is increasing and the field is constant and into the page, the flux is increasing into the page, therefore the induced current would generate a field pointing out of the page to fight this increase in flux into the page. By the right hand rule the current should be $\boxed{\text{counterclockwise}}$.

(b) Since the rods are moving at constant velocity, the total force on each rod must be zero. Because there is a current in the rods and the field is nonzero, they will experience a magnetic force similar to the first problem. By the previously calculated direction of current and the right hand rule the magnetic force on the left rod should be to the right and the magnetic force on the right rod should be to the left. The external force necessary to make the total force on the rods zero should be equal and opposite to the magnetic force on each rod. Since they both have the same current in the same field and are the same length, the magnitude of the magnetic force $|\vec{F}|$ is the same for both rods.

$$|\vec{F}| = F = IlB\sin\theta = IlB = \boxed{\frac{vl^2B^2}{R}}$$

Where $\theta = \frac{\pi}{2}$ is the angle between the direction of current $\vec{l}$ and the magnetic field $\vec{B}$, and the last equality is the result of plugging in part (a). An external force of this magnitude must be exerted $\boxed{\text{to the left on the left rod}}$ and $\boxed{\text{to the right on the right rod}}$.

(c) Since energy is conserved, the power being dissipated by the circuit must be the equal to the power being provided to the circuit. Since the external force provides all the power, calculate the power being dissipated by the circuit. Here the circuit is simply a resistor with resistance 2R, so the power being dissipated is given by the relationship (for a resistor resistance R) $P = IV = I^2R$.

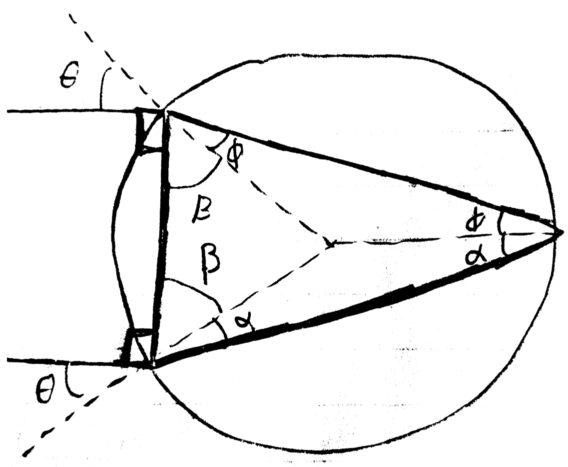
$$P = \left(\frac{Blv}{R}\right)^2 (2R) = \boxed{\frac{2B^2l^2v^2}{R}}$$

3. Ultimately use the geometry of the problem, the small angle approximation, Snell's law, and the law of reflection to solve for n_2/n_1.

(a) Since ϕ_1 and ϕ_2 form an isosceles triangle with the third vertex being the center of the sphere they must satisfy $\phi_1 = \phi_2 = \phi$.

$$\frac{\phi_1}{\phi_2} = \boxed{1}$$

(b) Use the remaining geometry to solve for θ/ϕ. According to this diagram with extra labels present $\beta = \frac{\pi}{2} - \theta$, and by the law of reflection, which is given explicitly in the geometry, $\alpha = \phi$. Finally, the sum of the interior angles in the bold triangle must add up to π.



$$\pi = 2\beta + 2\alpha + 2\phi = \pi - 2\theta + 4\phi$$
$$\Rightarrow \theta = 2\phi$$
$$\Rightarrow \frac{\theta}{\phi} = \boxed{2}$$

(c) Snell's law gives another relationship between θ and ϕ.

$$n_1 \sin\theta = n_2 \sin\phi$$

Since θ is assumed to be a small angle, ϕ will necessarily be small as well, so make the small angle approximation ($\sin x \to x$ when $x << 1$).

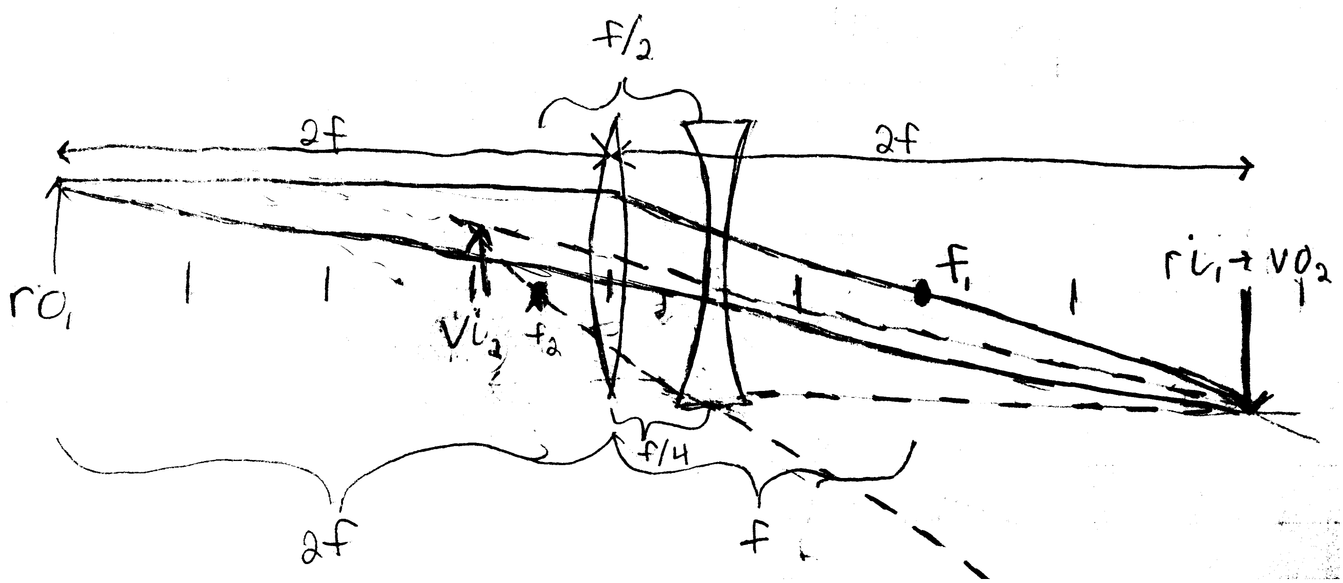
$$n_1 \sin\theta = n_2 \sin\phi \to n_1\theta = n_2\phi$$

$$\Rightarrow \frac{n_2}{n_1} = \frac{\theta}{\phi}$$

Then plug this into the result from part (b) to obtain the answer.

$$\frac{n_2}{n_1} = \boxed{2}$$

4. (a) See the following diagram, which is a reasonably accurate approximation of the correct answer. The solid rays are for tracing the first lens, then the image of the first lens becomes the object for the second lens, with dashed rays for the ray tracing of the second lens.



(b) Here we need to solve the thin lens equation for both stages of this two lens system and then find the relative position of the final image to the first lens d. s_1 and s_1' will be the object and image distances for the first lens $f_1 = f$ while s_2 and s_2' will be the object and image distance for the second lens $f_2 = -\frac{f}{2}$. First use the thin lens equation to solve the for s_1' given the object distance $s_1 = 2f$.

$$\frac{1}{s_1} + \frac{1}{s_1'} = \frac{1}{f_1} \Rightarrow \frac{1}{2f} + \frac{1}{s_1'} = \frac{1}{f}$$

$$\Rightarrow s_1' = 2f$$

This gives the real image show as ri_1 in the diagram. This image becomes the object for the second lens vo_2. Since it is to the right of the lens it is a virtual object, so $s_2 < 0$. Since the second lens is $f/4$ to the right of the first lens the geometry gives the following relationship between s_1' and s_2.

$$s_2 = -\left(s_1' - \frac{f}{4}\right) = -\frac{7f}{4}$$

Use this object distance in the thin lens equation for the second lens to solve for the final image position relative to second lens s_2'.

$$\frac{1}{s_2} + \frac{1}{s_2'} = \frac{1}{f_2} \Rightarrow -\frac{4}{7f} + \frac{1}{s_1'} = -\frac{2}{f}$$

$$\Rightarrow s_2' = -\frac{7f}{10}$$

Which means the final image is virtual and to the left of the second lens by a distance $7f/10$ at the location of vi_2 in the diagram. Since this distance is greater than the separation between the two lenses, the final image is to the left of the first lens as shown in the diagram. Use the geometry of the problem to find the distance d between the final image and the first lens.

$$d = |s_2'| - \frac{f}{4} = \frac{7f}{10} - \frac{f}{4} = \boxed{\frac{9f}{20}}$$

(c) To find the focal length f_c of a single lens at the location of the first lens that produces an image at the same location of this two lens system, plug the object distance $s_c = 2f$ and image distance $s_c' = -d$ into the thin lens equation. Here the signs of the distances are given by the sign convention for lenses. Solve for the focal length.

$$\frac{1}{s_c} + \frac{1}{s_c'} = \frac{1}{f_c} \Rightarrow \frac{1}{2f} - \frac{20}{9f} = \frac{1}{f_c}$$

$$f_c = \boxed{-\frac{18}{31f}}$$

5. For this problem there is a magnetic field produced by the current in the center that is canceled exactly by the magnetic field generated by a changing electric field in some circular region of radius a. To solve this problem, use the corrected form of Ampere's law.

$$\oint_{path} \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \epsilon_o \frac{\partial}{\partial t} \oint_{surface} \vec{E} \cdot d\vec{a}$$

(a) Since the field outside of the region of radius a is zero, use the corrected form of Ampere's law to solve for the necessary rate of change of $\vec{E}$ to make this true by choosing a circular path concentric with the region of radius $r > a$. $\vec{B}$ is zero along this entire path.

$$\oint_{path} \vec{B} \cdot d\vec{l} = 0$$

This path is bounded by a surface that covers the entire region of changing electric field, and that region has an area πa^2 with the electric field perpendicular to the surface.

$$\oint_{surface} \vec{E} \cdot d\vec{a} = E\pi a^2$$

This surface has a current I flowing through it, so $I_{enc} = I$. Write down Ampere's law making these substitutions.

$$0 = \mu_0 I + \mu_0 \epsilon_0 \frac{\partial}{\partial t} E\pi a^2$$

Now take the derivate with respect to time and simplify the equation.

$$0 = I + \epsilon_0 \pi a^2 \frac{\partial E}{\partial t}$$

$$\Rightarrow \frac{\partial E}{\partial t} = \boxed{-\frac{I}{\epsilon_0 \pi a^2}}$$

(b) Since $\frac{\partial E}{\partial t}$ is now known, use the corrected form of Ampere's law to solve for the magnetic field $\vec{B}(r)$ for $r < a$ by choosing a circular path of radius r. Now $\vec{B}(r)$ is nonzero, but is the same B magnitude at a constant radius by symmetry.

$$\oint_{path} \vec{B} \cdot d\vec{l} = B2\pi r$$

Now the path bounds a surface that encloses only part of the changing electric field up to radius r so the area is πr^2 and the electric field is perpendicular to the surface.

$$\oint_{surface} \vec{E} \cdot d\vec{a} = E\pi r^2$$

This surface also has a current I flowing through it still so $I_{enc} = I$. Substitute these expressions into the corrected form of Ampere's law.

$$B2\pi r = \mu_0 I + \mu_0 \epsilon_0 \frac{\partial}{\partial t} E\pi r^2$$

Take the derivative.

$$B2\pi r = \mu_0 I + \mu_0 \epsilon_0 \pi r^2 \frac{\partial E}{\partial t}$$

Plug in the result from part (a) and solve for $B(r)$.

$$B2\pi r = \mu_0 I - \mu_0 I \frac{r^2}{a^2}$$

$$\Rightarrow B(r) = \boxed{\frac{\mu_0 I}{2\pi r}\left(1 - \frac{r^2}{a^2}\right)}$$