Physics 8B Fall 2013
Solution to Final Problem 1

1. The energy of each level is given by

$$E_{n_x,n_y} = \frac{\pi^2\hbar^2}{2m_eL^2}\left(n_x^2+n_y^2\right) = E_0\left(n_x^2+n_y^2\right),$$

where $n_x = 1, 2, 3, \cdots$ and $n_y = 1, 2, 3, \cdots$ are two quantum numbers that specify the state (like n, l, m for the hydrogen atom).

(a) The state with smallest energy is obviously $(n_x, n_y) = (1, 1)$. Since each electron can spin up or spin down, two electrons can occupy the ground state $(n_x, n_y) = (1, 1)$ with different spins:

$$(n_x, n_y, m_s) = \left(1, 1, +\frac{1}{2}\right),$$
$$(n_x, n_y, m_s) = \left(1, 1, -\frac{1}{2}\right).$$

(Like the electron in a hydrogen atom, the two ground states are $(n, l, m, m_s) = (1, 0, 0, +1/2)$ and $(1, 0, 0, -1/2)$.) As a result, the first six states are

Electron	n_x	n_y	m_s
1	1	1	$+1/2$
2	1	1	$-1/2$
3	2	1	$+1/2$
4	2	1	$-1/2$
5	1	2	$+1/2$
6	1	2	$-1/2$

Notice that $(n_x, n_y) = (2, 1)$ and $(n_x, n_y) = (1, 2)$ are different states, although they have the same energy (like the degeneracy in hydrogen atom).

(b) The total energy of the six electrons is then given by

$$\begin{aligned} E_T &= E_0\left[\left(1^2+1^2\right)+\left(1^2+1^2\right)+\left(1^2+2^2\right)+\left(1^2+2^2\right)+\left(2^2+1^2\right)+\left(2^2+1^2\right)\right] \\ &= 24E_0, \end{aligned}$$

so

$$\boxed{\frac{E_T}{E_0} = 24}.$$

(You will still get full credit if your answer is consistent with that in part (a) even you did wrong there.)

(c) The next energy level is $(n_x, n_y) = (2, 2)$, so it requires

$$E_7 = E_0 \left(2^2 + 2^2\right) = 8E_0$$

energy to fill it, or

$$\boxed{\frac{E_7}{E_0} = 8}.$$

(Again you will get credit if your answer is consistent with that in part (a). By consistent I mean it's not a randomly guessed seemingly plausible sequence.)

(d) If the electrons did not have to obey Pauli exclusion principle, then all of them can occupy the ground state $(n_x, n_y) = (1, 1)$. In this case

$$\begin{aligned} E_T &= E_0 \left[\left(1^2 + 1^2\right) + \left(1^2 + 1^2\right) + \left(1^2 + 1^2\right) + \left(1^2 + 1^2\right) + \left(1^2 + 1^2\right) + \left(1^2 + 1^2\right)\right] \\ &= 12E_0, \end{aligned}$$

or

$$\boxed{\frac{E_T}{E_0} = 12}.$$

2. a) $\int_{-\infty}^{\infty} |\psi|^2 dx = A^2 \int_{-\infty}^{\infty} e^{-ax^2} dx = 1$, where $a = \frac{m\omega}{\hbar}$. So $1 = A^2 \sqrt{\frac{\pi}{a}}$

$\rightarrow A = \left(\frac{a}{\pi}\right)^{1/4} = \boxed{\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}}$

b) $\langle x \rangle = \int_{-\infty}^{\infty} x |\psi(x)|^2 dx = A^2 \int_{-\infty}^{\infty} x e^{-ax^2} dx = 0$

by symmetry (we are integrating an odd function over a symmetric range).

c) $\langle U \rangle = \int_{-\infty}^{\infty} U(x) |\psi(x)|^2 dx$ $\quad U(x) = \frac{1}{2} m\omega^2 x^2$

$= \frac{1}{2} m\omega^2 A^2 \int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2} m\omega^2 A^2 \frac{1}{2a} \sqrt{\frac{\pi}{a}}$

$= \frac{1}{2} m\omega^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{\hbar}{2m\omega} \left(\frac{\pi\hbar}{m\omega}\right)^{1/2} = \boxed{\frac{\hbar\omega}{4}}$,

which is half of the total energy of the ground state of the harmonic oscillator. This makes sense, because classically the energy shifts back and forth between potential and kinetic for the SHM.

3 Solution

a) $\lambda = \frac{h}{p}$

$p = m_e v$

$\lambda = \frac{h}{m_e v}$

$$\boxed{v = \frac{h}{m_e \lambda}}$$

b) $K = \frac{1}{2} m v^2$

$K = \frac{1}{2} m_e \frac{h^2}{m_e^2 \lambda^2}$

$$\boxed{K = \frac{1}{2} \frac{h^2}{m_e \lambda^2}}$$

c) Conservation of Energy

→ the potential energy of the electron is converted into Kinetic energy

$K = W$, where $W = eV$

$eV = \frac{h^2}{2 m_e \lambda^2}$

$$\boxed{V = \frac{h^2}{2 m_e e \lambda^2}}$$

PROBLEM 4

a. This part is purely a kinetic problem. In earth's frame,

Note that we've set all c's to be $\boxed{1}$ for conveniency, and that's why distance and time both share the unit "min", while the velocities are unitless.

After time $\Delta t = t_f - t_0$, the two spaceships meet and thus appear at the same position x_c.

$$x_c = x_A + v_A \Delta t = x_B + v_B \Delta t$$

$$\Rightarrow \Delta t = \frac{x_B - x_A}{v_A - v_B} = \frac{12\,\text{min} - 0}{\frac{4}{5} - \frac{3}{5}} = \boxed{60\ \text{min}}$$

b. We are still in the earth's frame, and the position of spaceship A when it catches up w/ spaceship B has been defined as x_c in the previous part.

$$x_c = x_A + v_A \Delta t = 0 + \frac{4}{5} \times 60\,\text{min} = \boxed{48\ \text{min}}$$

c. We provide two approaches to this part.

I. Lorentz Transformation

We are boosting towards the right while transforming from the earth's frame to the rest frame of spaceship A.

- There are three events in the scenario described in the problem

$(0,0) \to (x_A, t_0)$ the initial position of A in the earth frame

$(12\,\text{min}, 0) \to (x_B, t_0)$ the initial position of B in the earth frame.

$(48\,\text{min}, 60\,\text{min}) \to (x_c, t_f)$ the position where A & B meet.

After applying Lorentz transformations

$$\begin{cases} x' = \gamma_A (x - v_A t) \\ t' = \gamma_A (t - v_A x) \end{cases} \quad w/ \ \gamma_A = \frac{1}{\sqrt{1-v_A^2}} = \frac{5}{3}.$$

②

The events transfored into in A's frame

$(x_A', t_0') = (0, 0)$

$(x_B', t_0') = (\gamma_A x_B, 0 - \gamma v_A x_B) = (\frac{5}{3} \times 12\text{min}, -\frac{5}{3} \times \frac{4}{5} \times 12\text{min})$

$= (20\text{ min}, -16\text{ min})$

$(x_c', t_f') = (\gamma_A (x_c - v_A t_f), \gamma_A (t_f - v_A x_c))$

$= \left(\frac{5}{3} \times (48 - \frac{4}{5} \times 60), \frac{5}{3} \times (60 - \frac{4}{5} \times 48)\right) \text{min}$

$= (0, 36\text{ min})$

$x_c' = 0$ is expected since A never moved in its rest frame.

$\Rightarrow \boxed{\Delta x_A' = 0}$

$\underline{\Delta t_A'} = t_f' - t_0' = \boxed{36\text{ min}}$ by definition.

II. Direct Applications of Time Dilation formula

Since A never moved in its own frame, $\boxed{\Delta x_A' = 0}$

The two events: initial start point of A at $(0, 0)$,

A meeting B at $(x_c, t_f) = (48\text{ min}, 60\text{ min})$

happen at the same position in A's frame, so the time passed in A, $\Delta t_A'$, is the proper time between these two events.

$\Rightarrow \Delta t = t_f - t_0 = \gamma_A \Delta t_A'$ by applying the time dilation formula.

$\Rightarrow \Delta t_A' = \frac{\Delta t}{\gamma_A} = 60\text{min} / \frac{5}{3} = \boxed{36\text{ min}}$

An infinitely long hollow core wire inner radius a and outer radius b carries current I. To find the magnetic field, use Ampere's law with an amperian loop C which is a circle of radius r about the axis of the hollow wire.

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

Because the wire has cylindrical symmetry and the loop is about that symmetry axis $\vec{B} \cdot d\vec{l}$ must be a constant Bdl such that

$$\oint_C \vec{B} \cdot d\vec{l} = \oint_C B\, dl = B \oint_C dl = B2\pi r = \mu_0 I_{enc}.$$

Where the last integral was just the circumference of the circular loop of radius r. This gives us a simplified expression for the field at any radius once by calculating the current enclosed by the loop.

$$B(r) = \frac{\mu_0 I_{enc}}{2\pi r}$$

a) For the region $r < a$ there is no current passing through the loop so $I_{enc} = 0$. Plugging this into the simplified expression gives us that the magnetic field in this region is zero.

$$\boxed{B(r) = 0} \qquad r < a$$

b) For the region $r > b$ the loop surrounds the entire wire so $I_{enc} = I$. Plugging this into the simplified expression gives us that the magnetic field in this region is

$$\boxed{B(r) = \frac{\mu_0 I}{2\pi r}} \qquad r > b$$

c) For the region $a \leq r \leq b$ the loop encloses only part of the wire so calculate the amount of current enclosed for a particular radius r. The cross-sectional area A of the wire is

$$A = \pi b^2 - \pi a^2$$

so that the current density J of the wire is

$$J = \frac{I}{A} = \frac{I}{\pi\left(b^2 - a^2\right)}$$

The loop encloses a radius r but everything inside radius a is empty space with zero current, so the cross-sectional area of the wire enclosed by the loop A' is

$$A' = \pi r^2 - \pi a^2$$

Now multiply the current density of the wire by the cross-sectional area of the wire enclosed by the loop to find the enclosed current

$$I_{enc} = JA' = I\left(\frac{r^2 - a^2}{b^2 - a^2}\right)$$

and plug this into the simplified expression to find the magnetic field in this region.

$$\boxed{B(r) = \frac{\mu_0 I}{2\pi r}\left(\frac{r^2 - a^2}{b^2 - a^2}\right)} \qquad a \leq r \leq b$$

#6

a) $2nd = (m+1/2)\lambda$

$d = t$, take $m=0$

$2nt = \frac{\lambda}{2}$

$t = \frac{\lambda}{4n} = \frac{\lambda}{4 \cdot 2.5}$

$t = \frac{\lambda}{10}$

$t = \frac{700nm}{10} = \boxed{70nm}$

b) $t = t_0(1 - T/T_0)$

$\frac{t}{t_0} = 1 - T/T_0$

$T = T_0(1 - t/t_0)$

$T = 100s(1 - \frac{70}{100}) = 100(1-0.7) = 100 \cdot 0.3$

$\boxed{T = 30s}$

c) $\lambda = 400nm$

$t = \frac{\lambda}{10}$

$t = \frac{400nm}{10} = 40nm$

$T = T_0(1 - t/t_0)$

$T = 100s(1 - 40/100) = 100(1-0.4) = 100 \cdot 0.6$

$\boxed{T = 60s}$