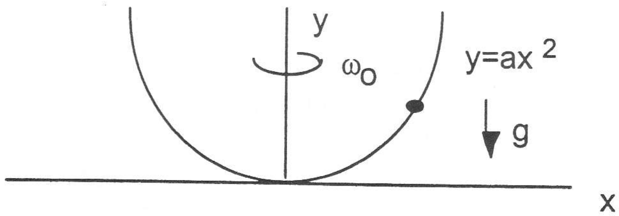
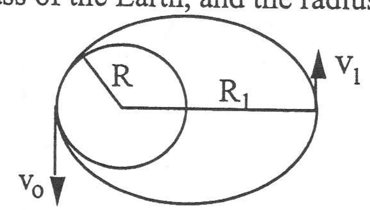
105 Final Exam

Answer all the questions. You must show the reasoning which leads to your answer to get full credit. Indicate the answers clearly and cross out work you feel is wrong.

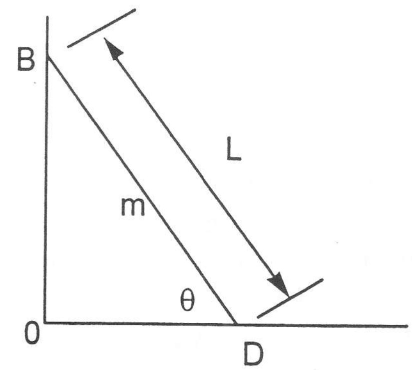
1. A wire of the shape of $y=ax^2$ is rotating around its vertical axis with an angular velocity ω_0, a point mass m is moving without friction on the wire under the gravitational force.
   (a) Write down the Lagrangian in terms of x and $\dot{x}$. (5 pts)
   (b) Derive the equation of motion. (5 pts)
   (c) At what value of ω_0, can the mass m stay at rest anywhere along the wire? (5 pts)



2. A satellite of mass m (m<<M) is launched from the Earth horizontally with a speed of v_0 ($v_0 > \sqrt{\frac{GM}{R}}$) into an elliptic orbit (see figure below).
   (a) What are the energy and the angular momentum of the satellite? (5 pts)
   (b) What is the farthest distance R_1 that the satellite can reach? (5pts)
   (c) What is the speed v_1 of the satellite at R_1? (5 pts)
   Write down the results in terms of G, M, m, R, and v_0. Here G, M, and R are the gravitational constant, the mass of the Earth, and the radius of the Earth, respectively.



3. A solid bar BD of mass m and length L is leant to the frictionless wall and frictionless ground with an initial angle $\theta=90^o$. The end B is constrained to move on the wall and end D is constrained to move on the ground. The bar is then released from rest. The inertia of rotation to the center of mass is $mL^2/12$. When the end B just reaches the ground,
   (a) show that the angular velocity is $\omega = \sqrt{\frac{3g}{L}}$. (5 pts)
   Hint: You may want to figure out the instant rotation axis A and use the law of conservation of energy.
   (b) show that the angular acceleration is $\dot{\omega} = \frac{3g}{2L}$ (5 pts)
   (c) what'is the normal force of the ground at end D? (5 pts)
   Hint: You may consider to write down the equation of motion to the center of mass rotation axis.



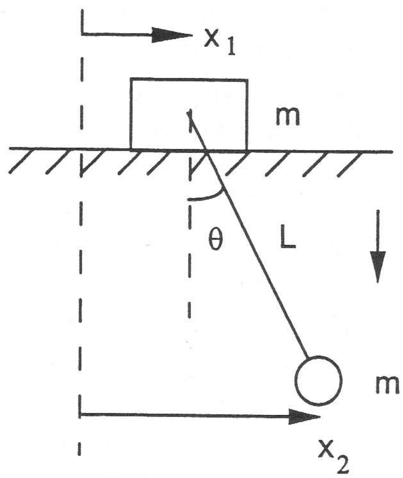
4. A massless rod of length L connects a point mass m with a block of mass m to form a pendulum. The block can move freely on a table without friction (see figure below). For small amplitude oscillations ($\theta<<1$),

(a) show that the Lagrangian of this system is $L=\frac{m\dot{x}_1^2}{2}+\frac{m\dot{x}_2^2}{2}-\frac{mg(x_2-x_1)^2}{2L}$. (5 pts)

(b) derive the equations of motion for x_1 and x_2. (5 pts)

(c) what are the normal mode frequencies? (5 pts)

(d) For each normal mode, find out the relation between x_1 and x_2. (5 pts)



5. Consider an infinitely long continuous string of linear mass density η with tension τ. A spring (spring constant K_0) is connected to the string at x=0. For a plane wave u(x,t)=Aexp(ikx-iωt) coming from the left, the reflection and transmission occur at x=0.

(a) Show that at x=0, $\tau\frac{\partial u(0^+,t)}{\partial x}-\tau\frac{\partial u(0^-,t)}{\partial x}-K_0u(0,t)=0$. (5pts)

(b) Find out the reflection and transmission coefficients R and T. (10 pts)
Hint: Check your results in the limits of $K_0\to 0$ and $K_0\to\infty$.