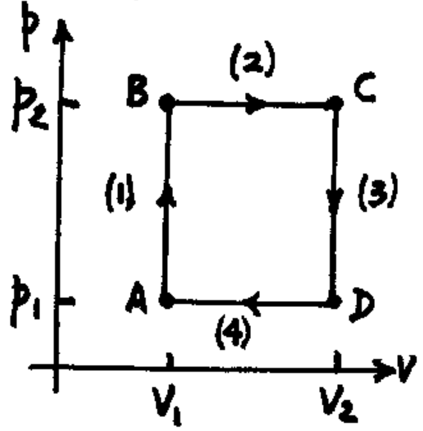
Physics 7B (Sec. 2) Midterm Exam #1 Feb. 25, 2003

You may use one (1) card, not larger than 3"×5" (both sides) as a memory aid, but no other papers, and no books. Exam = 220 points

(40) (1) Given one mole of an ideal monatomic gas undergoing the cyclic series of processes shown on the pV diagram. (a) Describe in words the process (1) which takes the gas from state A to state B. Describe also a method of realizing this process experimentally; (b) Do the same for process (2) taking the gas from state B to state C; (c) Calculate the total work W done by the gas during the cycle A → B → C → D → A; (d) From your answer to part (c), state whether positive work is done by the gas on the surroundings, or by the surroundings on the gas. Justify your answer; (e) Calculate the temperatures T_A, T_B, T_C, T_D of the gas in the states A, B, C, D; (f) Calculate the changes ΔU_1, ΔU_2, ΔU_3, ΔU_4 in the internal energy U of the gas in the processes (1), (2), (3), (4), in terms of pressures and volumes; (g) Calculate the heat Q_1, Q_2, Q_3, Q_4 in each of the processes, in terms of pressures and volumes; (h) From your answer to part (g), state whether heat energy enters or leaves the gas during each of the four processes; (i) Calculate the efficiency e of a heat engine based on this cycle; (j) Calculate the change ΔS in the entropy S of the gas during the cycle. [Each part = 4 points]



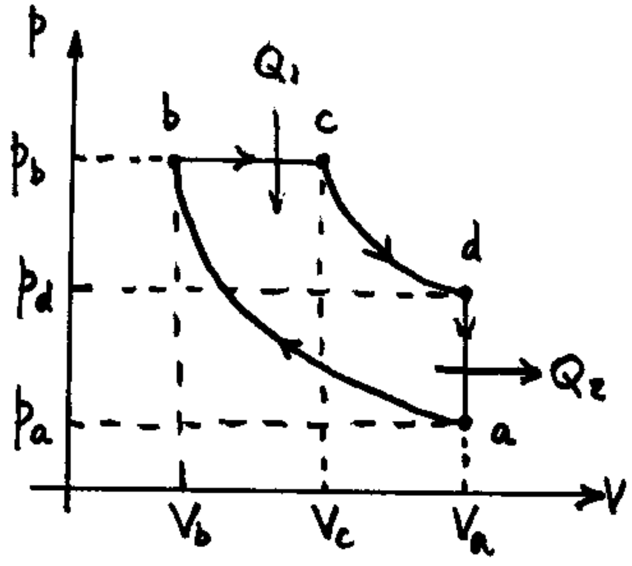
(continued →)

(30) (2) One mole of helium (an ideal gas) is at 120K with a volume of 0.10 m^3 at a pressure of $10^4\ \mathrm{Nm^{-2}}$. The gas undergoes a three-step process: (1) compression to a volume of 0.05 m^3 at a constant pressure of $10^4\ \mathrm{Nm^{-2}}$; (2) heating at a constant volume of 0.05 m^3 to a pressure of $2\times10^4\ \mathrm{Nm^{-2}}$; (3) reversible isothermal expansion to a volume of 0.10 m^3. (a) Draw a pV diagram showing the three steps of this process; (b) Calculate the temperature of the gas at the beginning and end of each step, and show the temperatures on the pV diagram; (c) Calculate the work done by the gas during the three-step process; (d) Calulate the change in the entropy of the gas during the three-step process [a=d=5; b=c=10]

(30) (3) One mole of an ideal monatomic gas, initially at volume V_0, pressure p_0, and temperature T_0, undergoes a reversible two-step process as follows: (1) the gas is heated at constant volume until its pressure is $2p_0$ and its temperature is T_1; (2) the gas then expands isothermally until its volume is V_1 and its pressure is again p_0. (a) Draw the pV diagram for this process, labeling all axes and points clearly; (b) Calculate the internal energy change ΔU_1, heat involved Q_1, and work and work W_1 done by the gas, for step (1) in terms of T_0 and constants; (c) Calculate ΔU_2, Q_2, and W_2 for step (2) in terms of T_0 and constants; (d) In the two-step process, does the gas do work on the surroundings, or do the surroundings do work on the gas? Justify your answer. [a=5, b=10, c=10, d=5 pts]

(continued→)

(45)(4) The pV diagram below shows a reversible quasi-static cycle in which paths cd and ab are adiabatic. The working substance for a heat engine using this cycle is an ideal gas. An amount of heat Q_1 enters the engine during path bc (at constant pressure p_b) and an amount of heat Q_2 leaves the engine during path da (at constant volume V_a). Show that the efficiency e of a heat engine using this cycle is



$$e = 1 - \left\{ \frac{(V_a/V_c)^{-\gamma} - (V_a/V_b)^{-\gamma}}{\gamma\left[(V_a/V_c)^{-1} - (V_a/V_b)^{-1}\right]} \right\}$$

$V_a^{-\gamma}\left(\left(\frac{1}{V_c^{-\gamma}}\right) - \left(\frac{1}{V_b^{-\gamma}}\right)\right)$

$= \frac{1}{V_a^{\gamma}}$ [illegible]

where $\gamma \equiv (c_p/c_v)$ for the working substance.

(30) (5) Consider n moles of a hypothetical <u>non-ideal</u> gas which obeys the equation of state $pV = nRT + \alpha p$, where α is a positive constant with dimensions of volume; α is small (but non-negligible) compared to any volume in this problem. Suppose that, for this gas, its internal energy depends only on temperature. The gas expands reversibly and isothermally from volume V_1 to volume V_2. (a) Calculate the change ΔS in the entropy S of the gas during the expansion; (b) Is ΔS positive or negative? Justify your answer with an explanation. [a = 20, b = 10 pts]

(continued ⟶)

(45)(b) Given a cylindrical pipe, of inner radius R_1 and outer radius R_2, made of a substance of thermal conductivity k; the length of the pipe is L. The pipe carries a hot liquid at temperature T_1 and the outer surface of the pipe is found to be at a (constant) temperature T_2, where $T_2 < T_1$. Calculate the constant rate (dQ/dt) of heat loss from the hot liquid to the outer surface.

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(1)(a) Process (1) is an increase in the pressure of the gas from p_1 to p_2 while the volume remains constant at V_1. To realize this process experimentally, place one mole of ideal gas (say, He) in a closed flask of volume V_1 and heat until the pressure increases from p_1 to p_2.

(b) Process (2) is an expansion of the gas from volume V_1 to volume V_2 at the constant pressure p_2. This can be realized experimentally by placing the closed flask of gas (with a moveable piston in it) in the state (p_2, V_1) and heating it. The volume will increase from V_1 to V_2 as the piston moves upward; continue until the volume is V_2.

(c) Total work W done by the gas during the four steps is $W = W_1 + W_2 + W_3 + W_4$

$W_1 = 0$ since process (1) is at constant volume V_1

$W_2 = \int_B^C p_2\,dV = p_2 \int_B^C dV = p_2(V_2 - V_1)$ since process (2) is at const. pressure p_2

$W_3 = 0$ since process (3) is at at constant volume V_2

$W_4 = \int_D^A p_1\,dV = p_1 \int_D^A dV = p_1(V_1 - V_2)$ since process (4) is at const. pressure p_1

$$W = p_2(V_2 - V_1) + p_1(V_1 - V_2) \Rightarrow W = (p_2 - p_1)(V_2 - V_1)$$

(d) Since $p_2 > p_1$ and $V_2 > V_1$, $(p_2 - p_1) > 0$ and $(V_2 - V_1) > 0$, so $W > 0$, meaning that (positive) work is done by the gas on the surroundings

(e) Gas is ideal, so $pV = nRT = RT$ (since $n = 1$) throughout), leading to

$$T_A = (p_1 V_1 / R);\ T_B = (p_2 V_1 / R);\ T_C = (p_2 V_2 / R);\ T_D = (p_1 V_2 / R)$$

(continued →)

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(1)[continued](f) For one mole of monatomic ideal gas, $U = (3/2)RT$, so

$$\Delta U_1 = (3/2)R(T_B - T_A) = (3/2)R(p_2V_1 - p_1V_1)(1/R) = (3/2)V_1(p_2 - p_1) > 0$$

$$\Delta U_2 = (3/2)R(T_C - T_B) = (3/2)R(p_2V_2 - p_2V_1)(1/R) = (3/2)p_2(V_2 - V_1) > 0$$

$$\Delta U_3 = (3/2)R(T_D - T_C) = (3/2)R(p_1V_2 - p_2V_2)(1/R) = (3/2)V_2(p_1 - p_2) < 0$$

$$\Delta U_4 = (3/2)R(T_A - T_D) = (3/2)R(p_1V_1 - p_1V_2)(1/R) = (3/2)p_1(V_1 - V_2) < 0$$

(g) From First Law, $\Delta U = Q - W \Rightarrow Q = \Delta U + W$

$$Q_1 = \Delta U_1 + W_1 = (3/2)V_1(p_2 - p_1) > 0$$

$$Q_2 = \Delta U_2 + W_2 = (3/2)p_2(V_2 - V_1) + p_2(V_2 - V_1) = \tfrac{5}{2}p_2(V_2 - V_1) > 0$$

$$Q_3 = \Delta U_3 + W_3 = (3/2)V_2(p_1 - p_2) < 0$$

$$Q_4 = \Delta U_4 + W_4 = (3/2)p_1(V_1 - V_2) + p_1(V_1 - V_2) = \tfrac{5}{2}p_1(V_1 - V_2) < 0$$

(h) Process (1): $Q_1 > 0 \Rightarrow$ heat energy enters gas
Process (2): $Q_2 > 0 \Rightarrow$ heat energy enters gas
Process (3): $Q_3 < 0 \Rightarrow$ heat energy leaves gas
Process (4): $Q_4 < 0 \Rightarrow$ heat energy leaves gas

(i) Efficiency $e = (W_{TOT}/Q_{IN}) = (W_{TOT}/[Q_1 + Q_2])$

$$e = \{(p_2 - p_1)(V_2 - V_1)/[\tfrac{3}{2}V_1(p_2 - p_1) + \tfrac{5}{2}p_2(V_2 - V_1)]\}$$

(j) Process is cyclic, gas begins and ends in state A, and entropy S is a state variable, so entropy change $\boxed{\Delta S = (S_A - S_A) = 0}$

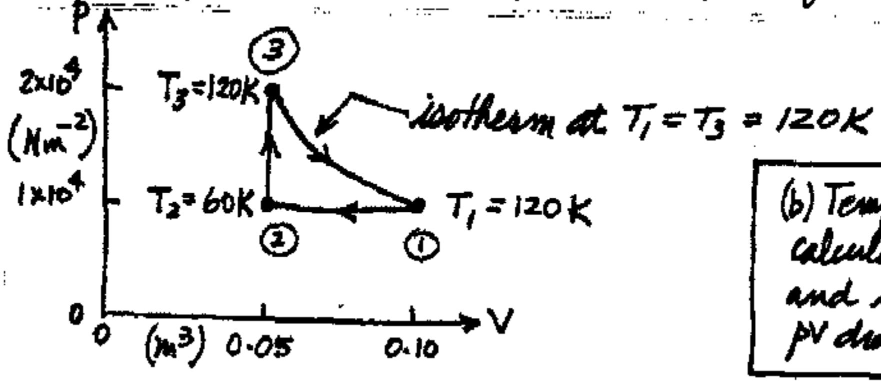
(2)(a) To draw a pV diagram, we need to identify the points which represent the various states of the gas:

Point 1: $p_1 = 10^4\,Nm^{-2}, V_1 = 0.10\,m^3, T_1 = 120K$

Point 2: $p_2 = 10^4\,Nm^{-2}, V_2 = 0.05\,m^3, T_2 = 60K$ (T_2 comes from $p_2V_2 = (1)RT_2$)

Point 3: $p_3 = 2\times10^4\,Nm^{-2}, V_3 = 0.05\,m^3, T_3 = 120K$ (T_3 comes from $p_3V_3 = (1)RT_3$)

Point 4: $p_4 = 10^4\,Nm^{-2}, V_4 = 0.10\,m^3, T_4 = 120K$ (where p_4 comes from $p_4V_4 = RT_4$ [isothermal step 3 → 4] so point 4 is same as point 1 and therefore the process is cyclic. The pV diagram is:



(b) Temperatures calculated above and shown on pV diagram

(c) Calculate work done by gas in each of three steps: $W = \int p\,dV$

Step 1: $p = p_1 = p_2 = 10^4\,Nm^{-2}$: $W_1 = \int_{0.10}^{0.05} p\,dV = p_1 \int dV = p_1(0.05 - 0.10)$

$W_1 = (10^4)(-5\times10^{-2})\,J.$

$\boxed{W_1 = -500\text{ Joules}}$ (-500 J.) of work done by gas means 500 J. of work done on gas

(continued →)

(2) [continued] Step 2: Volume change = 0 $\Rightarrow \boxed{W_2 = 0}$

Step 3: Isothermal expansion at $T = 120K$ from $V_3 = 0.05\,m^3$ to $V_4 = V_1 = 0.10\,m^3$. Gas is ideal so $p = (RT/V)$, and

$$W_3 = \int p\,dV = RT\int_{0.05}^{0.10}\frac{dV}{V} = RT\ln\left(\frac{0.10}{0.05}\right) = RT\ln 2$$

$$\boxed{W_3 = (8.31)(120)(0.693) = +691 \text{ Joules}}$$

Total work W done by gas in three-step process is $W = W_1 + W_2 + W_3$

$$\boxed{W = +191 \text{ Joules}}$$

Since $W > 0$, gas does 191 J. of work on surroundings

(d) Since process is cyclic, the initial and final states of the gas are the same, so entropy S_i in initial state 1 is same as entropy S_f in final state 4; entropy change in cyclic process is therefore

$$\boxed{\Delta S = (S_f - S_i) = 0}$$

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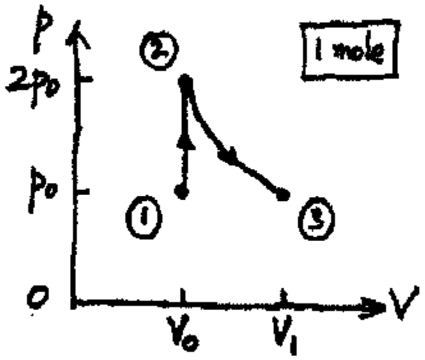
(3)(a) <u>To draw</u> the pV diagram, we need the <u>three points</u> defining the two-step process:

<u>Point ①</u>: p_0, V_0, T_0 $\rangle$ heating at constant volume (Step ①)
<u>Point ②</u>: $2p_0, V_0, T_1$ $\rangle$ isothermal expansion at temperature T_1 (Step ②)
<u>Point ③</u>: p_0, V_1, T_1



<u>(b) Step①</u>: $\Delta U_1 = Q_1 - W_1$; since $V = V_0 =$ const. $\boxed{W_1 = 0}$

$\Rightarrow \quad \Delta U_1 = Q_1$

Since this is an ideal monatomic gas, the internal energy $U = \frac{3}{2}RT$

so $\quad \Delta U_1 = \frac{3}{2}R(T_1 - T_0)$

<u>To find ΔU_1</u>, we need to find T_1. Since the gas is ideal,

$$p_0V_0 = RT_0 \;\; \underline{\text{and}} \;\; (2p_0)V_0 = RT_1 \quad \Longrightarrow \quad \boxed{T_1 = 2T_0}$$

so $\quad \boxed{\Delta U_1 = \frac{3}{2}R(2T_0 - T_0) = \frac{3}{2}RT_0}$

and $\quad \boxed{Q_1 = \Delta U_1 = \frac{3}{2}RT_0}$

(continued ⇒)

(3) [continued]

(c) Step ② : $\Delta U_2 = Q_2 - W_2$; process is isothermal at $T = T_1 = 2T_0$

Since the gas is ideal, U depends only on the temperature and the temperature is constant

$$\Delta U_2 = 0$$

Hence $Q_2 = W_2 = \int_{V_0}^{V_1} p\,dV$ where $p = RT(1/V)$

$$W_2 = \int_{V_0}^{V_1} RT_1 \frac{dV}{V} = RT_1 \ln\left(\frac{V_1}{V_0}\right) = (2RT_0)\ln\left(\frac{V_1}{V_0}\right)$$

To find V_1: For this isothermal process on an ideal gas,

$$(2p_0)V_0 = RT_1 \quad \text{and} \quad p_0 V_1 = RT_1$$

$$\boxed{V_1 = 2V_0} \quad \Longrightarrow (V_1/V_0) = 2$$

so $\boxed{W_2 = (2RT_0)\ln 2}$ and $\boxed{Q_2 = (2RT_0)\ln 2}$

(d) Total work W done by gas is

$$W = W_1 + W_2 = 0 + (2RT_0)\ln 2 = (2RT_0)\ln 2$$

Since $\ln 2 > 0$, $W > 0$

meaning gas does work on surroundings during the two-step process (that is, during the isothermal expansion).

(4) The efficiency e of the heat engine is given by

$$e = \frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1}$$

The only heat flows are Q_1 (in) and Q_2 (out) because steps cd and ab are adiabatic ($Q=0$). Call the temperature change of the gas in step bc (when heat Q_1 enters) by the label (ΔT_{bc}). Since step bc is at constant pressure, we have

$$Q_1 = nC_p(\Delta T_{bc}) \qquad (n = \text{\# of moles})$$

Similarly, in step da (when heat Q_2 leaves) at constant volume,

$$Q_2 = -nC_V(\Delta T_{da})$$

(Since heat leaves the gas, $\Delta T_{da} < 0$, the gas cools, and $Q_2 > 0$ is the magnitude of the amount of heat leaving the gas in step da.) Then

$$e = \left\{1 - \frac{(-nC_V\,\Delta T_{da})}{(nC_p\,\Delta T_{bc})}\right\} = \left\{1 + \frac{C_V\,\Delta T_{da}}{C_p\,\Delta T_{bc}}\right\} = \left\{1 + \frac{\Delta T_{da}}{\gamma(\Delta T_{bc})}\right\}$$

and $\gamma \equiv (C_p/C_V)$. The gas is ideal, so $pV = nRT$, leading to

$(nR)^{-1}p_b(V_c - V_b) = (\Delta T_{bc})$ and $(nR)^{-1}V_a(p_a - p_d) = (\Delta T_{da})$

Substituting these results into the expression for the efficiency gives

$$e = \left\{1 + \frac{V_a(p_a - p_d)}{\gamma p_b(V_c - V_b)}\right\} = \left\{1 - \frac{V_a(p_d - p_a)}{\gamma p_b(V_c - V_b)}\right\} < 0$$

(continued →)

(4) [continued] which can be rewritten as

$$e = \left\{ 1 - \frac{(p_d/p_b) - (p_a/p_b)}{\gamma\left[(V_c/V_a) - (V_b/V_a)\right]} \right\}$$

Since steps cd and ab are adiabatic, it is true that

$$p_c V_c^{\gamma} = p_d V_d^{\gamma} \quad \underline{\text{and}} \quad p_a V_a^{\gamma} = p_b V_b^{\gamma} \qquad (1)$$

Also, from pV diagram, $V_d = V_a$ and $p_c = p_b$, so, rewriting (1) as

$$\frac{p_d}{p_c} = \left(\frac{V_c}{V_d}\right)^{\gamma} \quad \underline{\text{and}} \quad \frac{p_a}{p_b} = \left(\frac{V_b}{V_a}\right)^{\gamma}$$

we get $\quad (p_d/p_b) = (V_c/V_a)^{\gamma} \quad \underline{\text{and}} \quad (p_a/p_b) = (V_b/V_a)^{\gamma}$

or $\quad (p_d/p_b) = (V_a/V_c)^{-\gamma} \quad \underline{\text{and}} \quad (p_a/p_b) = (V_a/V_b)^{-\gamma} \qquad (2)$

Substituting (2) into the expression for e, and noting that

$$(V_c/V_a) = (V_a/V_c)^{-1} \quad \underline{\text{and}} \quad (V_b/V_a) = (V_a/V_b)^{-1}$$

we obtain for the efficiency e

$$e = \left\{ 1 - \frac{(V_a/V_c)^{-\gamma} - (V_a/V_b)^{-\gamma}}{\gamma\left[(V_a/V_c)^{-1} - (V_a/V_b)^{-1}\right]} \right\} \qquad \text{where } \gamma \equiv (c_p/c_v)$$

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(5) (a) Process is reversible, so $\Delta S = \int dS = \int (dQ/T)$ and process is isothermal so, with T constant,

$$\Delta S = \frac{1}{T}\int dQ = (Q/T)$$

Since the internal energy U depends only on T, $\Delta U = 0$, so First Law gives us

$$\Delta U = 0 = Q - W \Longrightarrow Q = W$$

To calculate W, the work done by the gas during the isothermal expansion,

$$W = \int_{V_1}^{V_2} p\, dV = \int_{V_1}^{V_2} \left(\frac{nRT}{V-\alpha}\right) dV \quad \text{since } pV = nRT + \alpha p$$

Since T = constant,

$$W = nRT\int_{V_1}^{V_2} \frac{dV}{(V-\alpha)} = nRT\Big[\ln (V-\alpha)\Big]_{V_1}^{V_2} = nRT \ln\left(\frac{V_2-\alpha}{V_1-\alpha}\right) = Q$$

Then the entropy change $\Delta S = (Q/T)$ is

$$\boxed{\Delta S = nR \ln\left(\frac{V_2-\alpha}{V_1-\alpha}\right)} \qquad (n, R \text{ both positive})$$

(b) Since $V_2 > V_1$ (expansion), it is true that $(V_2-\alpha) > (V_1-\alpha)$ so

$$\left(\frac{V_2-\alpha}{V_1-\alpha}\right) > 1 \Longrightarrow \ln\left(\frac{V_2-\alpha}{V_1-\alpha}\right) > 0 \Longrightarrow \boxed{\Delta S > 0}$$

The entropy of the gas increases during this expansion from $V_1 \to V_2$.

(6) The basic differential equation for conductive heat transfer is

$$\frac{dQ}{dt} = -kA\frac{dT}{dx} \qquad (1)$$

where k is the thermal conductivity, A is the area across which heat is transferred, and (dT/dx) is the temperature gradient. Since we are dealing with a cylindrical pipe, we rewrite (1) in terms of polar coordinates as

$$\frac{dQ}{dt} = -kA\frac{dT}{dr} \qquad (2)$$

(end view; length L into paper)

where r is the radial coordinate shown. Construct a cylindrical shell of radius r, thickness dr, and length L, where $R_1 \le r \le R_2$. The surface area of the cylindrical shell is

$$A = 2\pi r L$$

(length L into paper)

so equation (2) for the rate (dQ/dt) of heat loss becomes

$$\frac{dQ}{dt} = -k(2\pi r L)\frac{dT}{dr}$$

where we need to evaluate the radial temperature gradient (dT/dr).

We are told that (dQ/dt) = C = constant, so the rate (dQ/dt) of heat flow across the area of any cylindrical shell of any radius r is the same. Therefore

$$\frac{dQ}{dt} = C = -k(2\pi r L)\frac{dT}{dr}$$

(continued →)

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(6) [continued] We integrate the differential equation above to get

$$C\,dr = -k(2\pi r L)\,dT$$

where $R_1 \leq r \leq R_2$ and $T_1 \leq T \leq T_2$, so

$$C\int_{R_1}^{R_2} \frac{dr}{r} = -2\pi k L \int_{T_1}^{T_2} dT$$

$$C \ln\left(\frac{R_2}{R_1}\right) = -2\pi k L(T_2 - T_1) = 2\pi k L(T_1 - T_2)$$

which can be solved for the constant C

$$C = \frac{2\pi k L(T_1 - T_2)}{\ln(R_2/R_1)}$$

Since we know that the rate of heat loss $(dQ/dt) = C$, we have

$$\boxed{\frac{dQ}{dt} = \frac{2\pi k L(T_1 - T_2)}{\ln(R_2/R_1)}}$$

Since $T_1 < T_2$, $(dQ/dt) < 0$, meaning heat is lost with time.