ME 185 Midterm #1

(Oct. 5, 2010)

1. (7 pts.) Recall that the gradient of a scalar field $\phi(\mathbf{x})$ may be represented in the form $\nabla\phi = \phi_{,i}\mathbf{e}_i$, where $\phi_{,i} = \partial\phi/\partial x_i$ and x_i are Cartesian coordinates. Also recall that the divergence of a vector field $\mathbf{v}(\mathbf{x})$ may be represented in the form $div\mathbf{v} = v_{i,i}$ where $v_i = \mathbf{e}_i \cdot \mathbf{v}$, and the curl may be represented in the form $curl\mathbf{v} = e_{ijk}v_{k,j}\mathbf{e}_i$.

Use these facts to demonstrate that, if $\mathbf{u} = curl\mathbf{v}$ and $\mathbf{w} = curl\mathbf{u}$, then

$$w_i = \psi_{,i} - v_{i,jj},$$

where $\psi = div\mathbf{v}$. This is the component representation of the identity

$$curl(curl\mathbf{v}) = grad(div\mathbf{v}) - \Delta\mathbf{v},$$

where $\Delta = div(grad)$ is the Laplacian operator. [Hint: $e_{kij}e_{kpq} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$.]

2. (8 pts.) Let $\varphi(x_1, x_2)$ be a function of only *two* Cartesian coordinates in a 3D Cartesian coordinate system $\{x_i\}$. Thus, φ does not depend on x_3. Let

$$\mathbf{v}(\mathbf{x}) = \mathbf{x} + \varphi\mathbf{e}_3,$$

where $\mathbf{x}$ is the 3D position vector. (a) Show that

$$\nabla\mathbf{v} = \mathbf{I} + \mathbf{e}_3 \otimes \nabla_2\varphi,$$

where ∇_2 is the two-dimensional gradient [i.e., $\nabla_2 f = (\mathbf{I} - \mathbf{e}_3 \otimes \mathbf{e}_3)\nabla f$, for any function $f(\mathbf{x})$. This is the projection of the 3D gradient onto the plane with unit normal $\mathbf{e}_3$]. (b) Show that $tr(\nabla\mathbf{v}) = 3$ and $\det(\nabla\mathbf{v}) = 1$. Therefore $\nabla\mathbf{v}$ is invertible. (c) Show that

$$(\nabla\mathbf{v})^{-1} = \mathbf{I} - \mathbf{e}_3 \otimes \nabla_2\varphi.$$

(d) Suppose $\varphi = g(\theta)$, where g is a given differentiable function and $\theta = \arctan(x_2/x_1)$. Show that $\nabla_2\varphi = r^{-1}g'(\theta)\mathbf{e}_\theta$, where $r = \sqrt{x_1^2 + x_2^2}$ and $\mathbf{e}_\theta = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2$.

3. (5 pts.) Consider a vector field $\mathbf{v}(\mathbf{x})$ defined by

$$\mathbf{v}(\mathbf{x}) = r\, f(r)\mathbf{e}_\theta,$$

where $f(r)$ is a given function (r and $\mathbf{e}_\theta$ are defined in #2(d) above). (a) Show that

$$\nabla\mathbf{v} = [\tfrac{d}{dr}(rf)]\mathbf{e}_\theta \otimes \mathbf{e}_r - f\mathbf{e}_r \otimes \mathbf{e}_\theta,$$

where $\mathbf{e}_r = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2$. (b) Write expressions for the symmetric, skew and deviatoric parts of $\nabla\mathbf{v}$. Specialize your results to the case $f(r) = \omega$, a constant.