ME 185 Midterm #1

(Oct. 18, 2007)

1. (10 pts.) Consider a cylindrical body occupying the reference configuration defined by $A < R < B$, $-L/2 < Z < L/2$, $0 \leq \theta < 2\pi$. Suppose the body is turned inside out (everted) so that after deformation it occupies a new cylindrical region (SEE FIGURE ON REVERSE). Thus, the deformation maps the material point with reference position

$$\mathbf{X} = R\mathbf{e}_r(\theta) + Z\mathbf{k}$$

to its final position

$$\mathbf{x} = r(R)\mathbf{e}_r(\theta) + z(Z)\mathbf{k},$$

where $a < r < b$ and $z(Z) = -Z$ (i.e., the cross-sectional plane $Z = L/2$ in the reference configuration is mapped to the plane $z = -L/2$ in the current configuration, etc.). Also, the inside of the reference cylinder is mapped to the outside of the deformed cylinder, and the outside is mapped to the inside. Thus, $r(A) = b$ and $r(B) = a$.

(a) Find the deformation gradient $\mathbf{F}$ assuming the function $r(R)$ to be known. What restrictions must be imposed on this function to ensure that the deformation is physically possible? Find $r(R)$ meeting the stated boundary conditions if the deformation is *isochoric.*

(b) Compute $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and obtain $\mathbf{U}$ *by inspection.* [Hint: $\mathbf{U}^2 = \mathbf{C}$]. Using your result, compute the rotation factor $\mathbf{R}$ in the polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$.

2. (10 pts.) Recall that the gradient of a scalar field $\phi(\mathbf{x})$ may be represented in the form $\nabla\phi = \phi_{,i}\mathbf{e}_i$, where $\phi_{,i} = \partial\phi/\partial x_i$ and x_i are Cartesian coordinates. Also recall that the divergence of a vector field $\mathbf{v}(\mathbf{x})$ may be represented in the form $div\mathbf{v} = v_{i,i}$ where $v_i = \mathbf{e}_i \cdot \mathbf{v}$, and the curl may be represented in the form $curl\mathbf{v} = e_{ijk}v_{k,j}\mathbf{e}_i$.

Use these facts to demonstrate that, if $\mathbf{u} = curl\mathbf{v}$ and $\mathbf{w} = curl\mathbf{u}$, then

$$w_i = \psi_{,i} - v_{i,jj},$$

where $\psi = div\mathbf{v}$. This is the component representation of the identity

$$curl(curl\mathbf{v}) = grad(div\mathbf{v}) - \Delta\mathbf{v},$$

where $\Delta = div(grad)$ is the Laplacian operator. [Hint: $e_{kij}e_{kpq} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$.]