Solution 1:

a) When system is heated from T_0 to T,

Mass of water remains constant, but volume of water changes. So, density of water changes.

$$Density\ \rho_{water} = Mass/Volume$$

$$\frac{\rho'_{water}}{\rho_o} = V_{water} \div V'_{water}$$

$$V'_{water} = V_{water}\left[1 + 3\beta\,(\mathrm{T} - \mathrm{T}_0)\right]$$

$$\frac{\rho'_{water}}{\rho_o} = \frac{1}{\left[1 + 3\beta\,(\mathrm{T} - \mathrm{T}_0)\right]}$$

$$\rho'_{water} = \frac{\rho_o}{\left[1 + 3\beta\,(\mathrm{T} - \mathrm{T}_0)\right]}$$

b) When system is heated from temperature T_0 to $T=T_B$

$$V'_{sphere} = V_{sphere}\left[1 + \beta\,(\mathrm{T}_B - \mathrm{T}_0)\right]$$

$$V'_{sphere} = V_0\left[1 + \beta\,(\mathrm{T}_B - \mathrm{T}_0)\right]$$

$$\rho'_{water} = \frac{\rho_o}{\left[1 + 3\beta\,(\mathrm{T}_B - \mathrm{T}_0)\right]}$$

Since sphere is suspended completely submerged, Buoyancy force=Gravity Force

$$\rho'_{water} V'_{sphere}\, g = Mg$$

$$\rho'_{water} V'_{sphere} = M$$

$$\frac{\rho_o}{\left[1 + 3\beta\,(\mathrm{T}_B - \mathrm{T}_0)\right]} \times V_0\left[1 + \beta\,(\mathrm{T}_B - \mathrm{T}_0)\right] = M$$

$$\rho_o V_0[1 + \beta\,(T_B - T_0)] = M\,[1 + 3\beta\,(T_B - T_0)]$$

$$T_B\,[\beta\rho_o V_0 - 3\beta M] = -\rho_o V_0 + \beta\rho_o V_0\,T_0 + M - 3\beta M T_0$$

$$T_B = \frac{\rho_o V_0[-1 + \beta T_0] + M\,[1 - 3\beta T_0]}{\beta[\rho_o V_0 - 3M]}$$

Problem 2

a)

The two sides undergo free expansion, and the total pressure will be the sum of the partial pressures. Thus:

$$P = \frac{2nRT_1}{3V} + \frac{nRT_2}{3V} = \frac{nR}{3V}(2T_1 + T_2)$$

Since the pressure does not change after the reaction, this means that:

$$T = \frac{P(3V)}{nR} = 2T_1 + T_2$$

b)

Since no work is done, by the first law $\Delta E_{int} = Q$. Thus:

$$Q = \frac{5}{2}nR(2T_1 + T_2) - \frac{3}{2}(2nRT_1 + nRT_2) = nR(2T_1 + T_2)$$

Problem 3:

(a).

If we define the heat flow into the water/ice mixture due to conduction to be positive, then over a time interval Δt, we have

$$\frac{Q_{wood}}{\Delta t} = \frac{k_w A}{l}(T_H - T_{water}) \quad (positive)$$

$$\frac{Q_{glass}}{\Delta t} = \frac{k_g A}{l}(T_L - T_{water}) \quad (negative)$$

Since the ice does not melt, the heat flow from wood balances the heat flow out to the glass at $T_H = T_F$. In degree Celsius, $T_{water} = 0℃$. Therefore, by setting $\frac{Q_{wood}}{\Delta t} + \frac{Q_{glass}}{\Delta t} = 0$, we have

$$\boxed{T_F = -\frac{k_g}{k_w} T_L \quad (℃)}$$

which makes sense since $T_L < 0$.

(b).

$$Q_{wood} + Q_{glass} = m_{ice} L_{ice}$$

$$T_H = 2T_F = -2\frac{k_g}{k_w} T_L$$

Therefore,

$$\frac{m_{ice} L_{ice}}{\Delta t} = \frac{Q_{wood}}{\Delta t} + \frac{Q_{glass}}{\Delta t} = \frac{k_w A}{l} T_H + \frac{k_g A}{l} T_L$$

$$= \frac{k_w A}{l}\left(-2\frac{k_g}{k_w} T_L\right) + \frac{k_g A}{l} T_L$$

$$= -\frac{k_g A}{l} T_L$$

$$\boxed{\Delta t = -\frac{m_{ice} L_{ice} l}{k_g A T_L}}$$

Problem 4.

(a).

$$V(T) = V_0 \left(\frac{T}{T_a}\right)^s \quad \Rightarrow \quad T = T_a \left(\frac{V}{V_0}\right)^{\frac{1}{s}}$$

$$P_a V_a = NkT_a \quad \Rightarrow \quad Nk = \frac{P_a V_a}{T_a} = \frac{P_0 V_0}{T_a}$$

$$\Rightarrow \quad PV = NkT = \left(\frac{P_0 V_0}{T_a}\right) T_a \left(\frac{V}{V_0}\right)^{\frac{1}{s}} = P_0 V_0 \left(\frac{V}{V_0}\right)^{\frac{1}{s}}$$

$$P(V) = P_0 \left(\frac{V}{V_0}\right)^{\frac{1}{s}-1} = \alpha V^{\frac{1}{s}-1}$$

where $\alpha = P_0 \left(\frac{1}{V_0}\right)^{\frac{1}{s}-1}$ is a constant.

For $P(V) = \alpha V^{\frac{1}{s}-1}$ to be concave up as is shown in the PV diagram, $\frac{1}{s} - 1 > 1$, therefore,

$$0 < s < \frac{1}{2} < 1$$

$$\boxed{s < 1}$$

(b).

From $P = P_0 \left(\frac{V}{V_0}\right)^{\frac{1}{s}-1}$

$$W_{ab} = \int_a^b P dV = \int_{V_a}^{V_b} P_0 \left(\frac{V}{V_0}\right)^{\frac{1}{s}-1} dV$$

Let $u = \left(\frac{V}{V_0}\right)$, then $dV = V_0 du$

$$W_{ab} = P_0 V_0 \int_1^2 u^{\frac{1}{s}-1} du = P_0 V_0 \left[s u^{\frac{1}{s}}\right]_1^2 = P_0 V_0 \left(s \left(2^{\frac{1}{s}} - 1\right)\right)$$

$$W_{bc} = 0$$

$$W_{ca} = P_0 (V_0 - 2V_0) = -P_0 V_0$$

$$\boxed{W_{cycle} = W_{ab} + W_{bc} + W_{ca} = P_0 V_0 \left(s \left(2^{\frac{1}{s}} - 1\right) - 1\right)}$$

(c).

$$E_a = \frac{5}{2}NkT_a = \frac{5}{2}P_0V_0$$
$$E_b = \frac{5}{2}NkT_b = \frac{5}{2}NkT_a 2^{\frac{1}{s}} = \frac{5}{2}\cdot 2^{\frac{1}{s}}P_0V_0$$
$$E_c = \frac{5}{2}NkT_c = \frac{5}{2}Nk(2T_a) = 5P_0V_0$$

$$Q_{ab} = \Delta E_{ab} + W_{ab} = \left(\frac{5}{2}\cdot 2^{\frac{1}{s}}P_0V_0 - \frac{5}{2}P_0V_0\right) + P_0V_0\left(s\left(2^{\frac{1}{s}} - 1\right)\right)$$
$$= P_0V_0\left(\frac{5}{2} + s\right)\left(2^{\frac{1}{s}} - 1\right) > 0$$
$$Q_{bc} = \Delta E_{bc} + W_{bc} = 5P_0V_0 - \frac{5}{2}\cdot 2^{\frac{1}{s}}P_0V_0 = 5\left(1 - 2^{\frac{1}{s}-1}\right)P_0V_0 < 0$$
$$Q_{ca} = \Delta E_{ca} + W_{ca} = \left(\frac{5}{2}P_0V_0 - 5P_0V_0\right) + (-P_0V_0) = -\frac{7}{2}P_0V_0 < 0$$

A faster way to see why Q_{bc} is negative is to observe that $\Delta E_{bc} = E_c - E_b < 0$ because $T_b > T_c$ from the PV diagram.

Therefore,

$$Q_{in} = Q_{ab} = P_0V_0\left(\frac{5}{2} + s\right)\left(2^{\frac{1}{s}} - 1\right)$$

From the PV diagram, we know that $W_{cycle} > 0$, thus it is a heat engine. Therefore,

$$\boxed{e = \frac{W_{cycle}}{Q_{in}} = \frac{s\left(2^{\frac{1}{s}} - 1\right) - 1}{\left(\frac{5}{2} + s\right)\left(2^{\frac{1}{s}} - 1\right)}}$$

Problem 5

a)

Method 1

Use the first law, the given relation of $dQ = \frac{1}{2}dW$ and the definition of internal energy to find an expression between P, V and T

$$\begin{aligned} dE &= dQ - dW \\ dE &= -PdV + \frac{1}{2}PdV \\ dE &= -\frac{1}{2}PdV \\ 3nRdT &= -PdV \end{aligned}$$

We integrate the left hand side from T_a to T_b

$$\int_{T_a}^{T_b} 3nRdT = 3nR(T_b - T_a)$$

Using the relationship $PV^\beta = const.$ given

$$P(V) = P_a \left(\frac{V_a}{V}\right)^\beta$$

Now use this to integrate the pressure side of the equation

$$-\int_{V_a}^{V_b} PdV = -\int_{V_a}^{V_b} P_a \left(\frac{V_a}{V}\right)^\beta dV = -\frac{P_a V_a^\beta}{1-\beta}(V_b^{1-\beta} - V_a^{1-\beta}).$$

Now we'll write this in terms of T, to better compare with the other side of the equation

$$\begin{aligned} -\int_{V_a}^{V_b} PdV &= -\frac{1}{1-\beta}(P_a V_a^\beta V_b^{1-\beta} - P_a V_a) \\ &= -\frac{1}{1-\beta}(P_b V_b^\beta V_b^{1-\beta} - P_a V_a) \\ &= -\frac{1}{1-\beta}(P_b V_b - P_a V_a) \\ &= -\frac{1}{1-\beta}(nRT_b - nRT_a) \end{aligned}$$

This must equal $3nR(T_b - T_a)$, so this means:

$$\begin{aligned} 3 &= \frac{1}{\beta - 1} \\ \beta &= \frac{4}{3} \end{aligned}$$

Method 2

Use the first law, the given relation of $dQ = \frac{1}{2}dW$ and the definition of internal energy to find an expression between P, V and T

$$\begin{aligned} dE &= dQ - dW \\ dE &= -PdV + \frac{1}{2}PdV \\ dE &= -\frac{1}{2}PdV \\ 3nRdT &= -PdV \end{aligned}$$

Use ideal gas law to eliminate P and arrive at an integrable expression

$$\begin{aligned} 3nRdT &= -\frac{nRT}{V}dV \\ 3\frac{dT}{T} &= -\frac{dV}{V} \end{aligned}$$

Integrate both sides and do some algebra

$$\begin{aligned} -3\int_{T_a}^{T_b} \frac{dT}{T} &= \int_{V_a}^{V_b} \frac{dV}{V} \\ -3\ln\left(\frac{T_b}{T_a}\right) &= \ln\left(\frac{V_b}{V_a}\right) \\ \left(\frac{T_b}{T_a}\right)^{-3} &= \frac{V_b}{V_a} \\ T_a^3 V_a &= T_b^3 V_b \end{aligned}$$

Now rewrite this in terms of P and V

$$\begin{aligned} T^3 V &= const. \\ \left(\frac{PV}{nR}\right)^3 V &= const. \\ P^3 V^4 &= const. \\ PV^{4/3} &= const. \end{aligned}$$

So, we get that $\beta = 4/3$.

b)

Using the definition of entropy,

$$S = \int_a^b \frac{dQ}{T} = \int_a^b \frac{dW}{2T} = \int_{V_a}^{V_b} \frac{PdV}{2T}$$

Using the ideal gas law,

$$S = \int_{V_a}^{V_b} \frac{nRdV}{2V} = \frac{nR}{2} \ln\left(\frac{V_b}{V_a}\right)$$

To put this in terms of T, use the ideal gas law with $PV^{4/3} = const.$ to get that $T_a^3 V_a = const.$ Thus,

$$S = \frac{3nR}{2} \ln\left(\frac{T_a}{T_b}\right).$$