Physics 137A

Lecture 2 ⋄ Spring 2014
University of California at Berkeley

Final Exam

May 16, 2014, 3-6pm, 2 LeConte
6 problems ⋄ 180minutes ⋄ 100points

Problem 1 ⋄ TWO SPIN-$\frac{1}{2}$ PARTICLES IN A SINGLET STATE *10points*

Consider a lab with with two experimenters: you and your favorite lab partner, studying a system of two distinguishable spin-$\frac{1}{2}$ particles is in a spin-singlet state – the state with total angular momentum eigenvalue 0 given by:

$$|\chi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \Big).$$

You are specializing in measuring the spin components of one of the particles (S_{1x}, S_{1z}, and so on) while your lab partner specializes in measuring the spin components of the other particles (S_{2x}, S_{2z}, and so on).

⋄ A ⋄ What is the probability for you to measure S_{1z} obtaining $\frac{\hbar}{2}$ if your lab partner makes no measurement?

⋄ B ⋄ Now the experiment is repeated but this time you are measuring S_{1x}. What is the probability you obtain $\frac{\hbar}{2}$ if your lab partner is still not doing anything?

⋄ C ⋄ Finally, the experiment is repeated for the third time and this time your lab partner decides to contribute: makes a measurement of S_{2z} and obtains $\frac{\hbar}{2}$. Then you get to work. What do you expect to be the outcome of your measurement if you measure S_{1z}? How about if you measure S_{1x}?

Problem 2 ⋄ CLEBSH-GORDAN COEFFICIENTS *30points*

Consider a spin-$\frac{1}{2}$ particle in a state with orbital angular momentum $l = 1$. The goal of this problem is to calculate the Clebsh-Gordan coefficients that allow us to construct the states of definite total angular momentum $\hat{J}$ from the simultaneous eigenstates of spin and orbital angular momentum. Label the states in the "old basis" (the eigenstates of $\hat{L}^2$, $\hat{L}_z$, $\hat{S}^2$, and $\hat{S}_z$) by $|l\ m_l\ s\ m_s\rangle$ and the states in the "new basis" (eigenstates of $\hat{J}^2$ and $\hat{J}_z$) by $|j\ m_j\rangle$.

⋄ A ⋄ Write all the states in the "new basis" in terms of states in the "old basis" and label the non-zero coefficients. How many non-zero coefficients is there?

⋄ B ⋄ Determine all these coefficients. For example, you can do this by starting with the state with maximum j and m_j and using lowering operators as well as orthonormality repeatedly.

⋄ C ⋄ What is the expectation value of $\hat{L}_z$ in the state with the lowest possible value of j and $m_j = j$? What is the expectation value of $\hat{S}_z$ in this state?

Problem 3 ⋄ FERMION GAS IN A HARMONIC TRAP *15points*

Consider a very large number N of noninteracting electrons of mass m_e.

⋄ A ⋄ The electrons are confined by a one-dimensional harmonic oscillator potential $V(x) = \frac{1}{2}m_e\omega^2x^2$. What is the value of the ground state energy? What is the value of the Fermi energy?

⋄ B ⋄ Now the electrons are confined to a three-dimensional version of this trap by the potential $V(\vec{r}) = \frac{1}{2}m_e\omega^2r^2$. What is the value of the Fermi energy for this system?

Problem 4 ⋄ VIRIAL THEOREM *15points*

⋄ A ⋄ Prove the virial theorem in one dimension, i.e. show that the expectation value of kinetic energy T in a *stationary* state relates to potential energy as:

$$2\langle T\rangle = \left\langle x\frac{dV}{dx}\right\rangle.$$

⋄ B ⋄ Show how this theorem generalizes to three dimensional space.

Problem 5 ◇ A STATIONARY STATE OF THE HARMONIC OSCILLATOR *15points*

A particle is in the n^{th} stationary state of the harmonic oscillator $|n\rangle$.

◇ A ◇ Find expectation values of $\langle x\rangle$ and $\langle x^2\rangle$.
◇ B ◇ Find expectation values of $\langle p\rangle$ and $\langle p^2\rangle$.
◇ C ◇ Check that uncertainty principle is satisfied.
◇ D ◇ Find expectation values of kinetic and potential energy and check that the virial theorem is satisfied.

Problem 6 ◇ STATES AND OPERATORS FOR A SPIN-1 PARTICLE *15points*

Consider a spin-1 particle with the usual basis states $\{|1\rangle, |0\rangle, |-1\rangle\}$ of eigenvectors of the $\hat{S}_z$, the z-component of spin, defined by $\hat{S}_z |m\rangle = m\hbar |m\rangle$. We can define three normalized states $|x\rangle$, $|y\rangle$, $|z\rangle$ by $\hat{S}_x |x\rangle = 0$, $\hat{S}_y |y\rangle = 0$, and $\hat{S}_z |z\rangle = 0$.

◇ A ◇ Express the states $|x\rangle$, $|y\rangle$, $|z\rangle$ in the basis $\{|1\rangle, |0\rangle, |-1\rangle\}$ and then show that they are mutually orthogonal (and therefore these three states are a good orthonormal basis in its own right.)
◇ B ◇ Define an operator $\hat{Q} = a|x\rangle\langle x| + b|y\rangle\langle y| + c|z\rangle\langle z|$, with a, b, and c all different real numbers. List eigenstates and corresponding eigenvalues of this operator.
◇ C ◇ Calculate $\langle 1|\hat{Q}|-1\rangle$ and hence show that $\hat{Q}$ is *not* an operator of the form $\hat{Q} = \vec{B}\cdot\hat{\vec{S}}$ for any magnetic field $\vec{B}$.
◇ D ◇ Explain what your conclusion in part C says about the possibility of designing a non-uniform $\vec{B}$ for a Stern-Gerlach experiment that would allow you to distinguishing the states $|x\rangle$, $|y\rangle$, and $|z\rangle$.

You might find these matrix representations for $s = 1$ spin operators (in the basis $\{|1\rangle, |0\rangle, |-1\rangle\}$) useful:

$$S_x = \sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \sqrt{2}\hbar \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Mathematical Formulas

Trigonometry:

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Gradient operator:

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\phi}$$

Laplace operator:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\right)$$

Integrals:

$$\int x\sin(ax)\,dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax)$$

$$\int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$$

Exponential integrals:

$$\int_0^\infty x^n e^{-x/a}\,dx = n!\,a^{n+1}$$

Gaussian integrals:

$$\int_0^\infty x^{2n} e^{-x^2/a^2}\ dx = \sqrt{\pi}\ \frac{(2n)!}{n!}\ \left(\frac{a}{2}\right)^{2n+1}$$

$$\int_0^\infty x^{2n+1} e^{-x^2/a^2}\ dx = \frac{n!}{2}\ a^{2n+2}$$

Integration by parts:

$$\int_a^b f \frac{dg}{dx}\ dx = -\int_a^b \frac{df}{dx} g\ dx + fg\Big|_a^b$$

Fundamental Equations

Schrödinger equation:

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$$

Time-independent Schrödinger equation:

$$\hat{H}\psi = E\psi, \quad \Psi = \psi e^{-iEt/\hbar}$$

Hamiltonian operator:

$$\hat{H} = \hat{T} + \hat{V} = \frac{\hat{p}^2}{2m} + V = -\frac{\hbar^2}{2m}\nabla^2 + V$$

Position and momentum representations:

$$\langle x \,|\, p\rangle = \tfrac{1}{\sqrt{2\pi\hbar}} exp(\tfrac{ipx}{\hbar}), \quad \psi(x) = \langle x \,|\, \psi\rangle\,, \quad \phi(p) = \langle p \,|\, \phi\rangle\,, \quad \langle x \,|\, \hat{p} \,|\, \psi\rangle = -i\hbar \tfrac{d}{dx}\psi(x)$$

Momentum operator:

$$\hat{p}_x = -i\hbar \tfrac{\partial}{\partial x}, \quad \hat{p}_y = -i\hbar \tfrac{\partial}{\partial y}, \quad \hat{p}_z = -i\hbar \tfrac{\partial}{\partial z}$$

Time dependence of an expectation value:

$$\frac{d\langle \hat{Q}\rangle}{dt} = \frac{i}{\hbar}\left\langle [\hat{H}, \hat{Q}]\right\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t}\right\rangle$$

Generalized uncertainty principle:

$$\sigma_A \sigma_B \geq \left| \frac{1}{2i}\left\langle [\hat{A}, \hat{B}]\right\rangle \right|$$

Canonical commutator:

$$[\hat{x}, \hat{p}_x] = i\hbar, \quad [\hat{y}, \hat{p}_y] = i\hbar, \quad [\hat{z}, \hat{p}_z] = i\hbar$$

Angular momentum:

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z, \quad [\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x, \quad [\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$$

Raising and lowering operator for angular momentum:

$$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y, \quad [\hat{L}_+, \hat{L}_-] = 2\hbar \hat{L}_z, \quad \hat{L}_\pm \,|l, m\rangle = \hbar\sqrt{l(l+1) - m(m \pm 1)}\,|l, m \pm 1\rangle$$

Raising and lowering operator for harmonic oscillator:

$$\hat{a}_\pm = \tfrac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} \mp i\hat{p}), \quad [\hat{a}_-, \hat{a}_+] = 1, \quad \hat{a}_+\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \quad \hat{a}_-\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

Pauli matrices for spin-$\frac{1}{2}$ particle:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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Problem 1

A. $|z\rangle$ is a linear combination of $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$.

The probability that the measurement would yield $+\hbar/2$ for $\hat{S}_{1z}$ is the probability that the particle is in $|\uparrow\downarrow\rangle$ state, which is $\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$.

B.

$$\left(\begin{array}{l}\text{probability of} \\ \text{being in state } |\uparrow\downarrow\rangle\end{array}\right) \times \left(\begin{array}{l}\text{probability that } |\uparrow\rangle_1 \\ \text{is in } |\hat{S}_x = \hbar/2\rangle_1 \text{ state}\end{array}\right) = \frac{1}{4}$$

$$\left(\begin{array}{l}\text{probability of} \\ \text{being in state } |\downarrow\uparrow\rangle\end{array}\right) \times \left(\begin{array}{l}\text{probability that } |\downarrow\rangle_1 \\ \text{is in } |\hat{S}_x = \hbar/2\rangle \text{ state}\end{array}\right) = \frac{1}{4}$$

$$\Sigma = \underline{1/2}$$

C. The wave function collapses to $|\downarrow\uparrow\rangle$ state. Then S_{1z} measurement will always yield $-\hbar/2$ and S_x will yield $\pm\hbar/2$ with 1/2 probability each.

PROBLEM 2

A.

New basis	Old basis
$\lvert 3/2 \; 3/2 \rangle$	$\lvert 1\,1\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle$
$\lvert 3/2 \; 1/2 \rangle$	$\lvert 1\,1\,\tfrac{1}{2}\,\tfrac{-1}{2}\rangle$, $\lvert 1\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle$
$\lvert 3/2 \; -1/2 \rangle$	$\lvert 1\,0\,\tfrac{1}{2}\,\tfrac{-1}{2}\rangle$, $\lvert 1\,-1\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle$
$\lvert 3/2 \; -3/2 \rangle$	$\lvert 1\,-1\,\tfrac{1}{2}\,\tfrac{-1}{2}\rangle$
$\lvert \tfrac{1}{2} \; \tfrac{1}{2} \rangle$	$\lvert 1\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle$, $\lvert 1\,1\,\tfrac{1}{2}\,\tfrac{-1}{2}\rangle$
$\lvert \tfrac{1}{2} \; \tfrac{-1}{2} \rangle$	$\lvert 1\,0\,\tfrac{1}{2}\,\tfrac{-1}{2}\rangle$, $\lvert 1\,-1\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle$

The states in the new basis are linear combinations of the states in the old basis that satisfy $m_j = m_\ell + m_s$.

B. $S_-\lvert \tfrac{3}{2}\,\tfrac{3}{2}\rangle = \sqrt{\tfrac{3}{2}\cdot\tfrac{5}{2}-\tfrac{3}{2}\cdot\tfrac{1}{2}}\;\lvert \tfrac{3}{2}\,\tfrac{1}{2}\rangle$

$(S_{1-}+S_{2-})\lvert 1\,1\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \sqrt{1\cdot 2-1\cdot 0}\;\lvert 1\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{2}\cdot\tfrac{3}{2}+\tfrac{1}{2}\cdot\tfrac{1}{2}}\;\lvert 1\,1\,\tfrac{1}{2}\,\tfrac{-1}{2}\rangle$

$= \sqrt{2}\,\lvert 1\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle + 1\,\lvert 1\,1\,\tfrac{1}{2}\,\tfrac{-1}{2}\rangle$

Thus $\lvert \tfrac{3}{2}\,\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,\lvert 1\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}\,\lvert 1\,1\,\tfrac{1}{2}\,\tfrac{-1}{2}\rangle$

Similarly, $\lvert \tfrac{3}{2}\,\tfrac{-1}{2}\rangle = \tfrac{1}{\sqrt{3}}\lvert 1\,-1\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,\lvert 1\,0\,\tfrac{1}{2}\,\tfrac{-1}{2}\rangle$ by repeating above procedure.

and $\lvert \tfrac{3}{2}\,\tfrac{-3}{2}\rangle = \lvert 1\,-1\,\tfrac{1}{2}\,\tfrac{-1}{2}\rangle$.

Now $\lvert \tfrac{1}{2}\,\tfrac{1}{2}\rangle$ has to be orthogonal to $\lvert \tfrac{3}{2}\,\tfrac{1}{2}\rangle$, so we can choose

$$\lvert \tfrac{1}{2}\,\tfrac{1}{2}\rangle = \tfrac{-1}{\sqrt{3}}\lvert 1\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,\lvert 1\,1\,\tfrac{1}{2}\,\tfrac{-1}{2}\rangle$$

Again, using $S_- = S_{1-}+S_{2-}$, we obtain $\lvert \tfrac{1}{2}\,\tfrac{-1}{2}\rangle = \tfrac{1}{\sqrt{3}}\lvert 1\,0\,\tfrac{1}{2}\,\tfrac{-1}{2}\rangle + - \sqrt{\tfrac{2}{3}}\,\lvert 1\,-1\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle$

PROBLEM 2

C. $$|\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \frac{-1}{\sqrt{3}}\,|1\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|1\,1\,\tfrac{1}{2}\,\tfrac{-1}{2}\rangle$$

$$\langle \hat{L}_z \rangle = (\tfrac{1}{3})(0) + (\tfrac{2}{3})(\hbar) = \tfrac{2}{3}\hbar$$

$$\langle \hat{S}_z \rangle = (\tfrac{1}{3})(\tfrac{\hbar}{2}) + (\tfrac{2}{3})(\tfrac{-\hbar}{2}) = -\frac{\hbar}{6}$$

PROBLEM 3

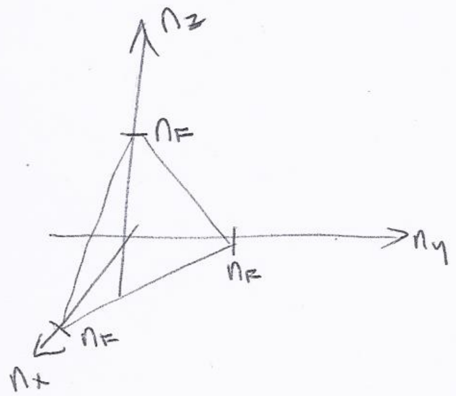
A. Electron is spin-1/2 fermion, so only two electrons can occupy each energy level. 1-D Harmonic Oscillator spectrum: $E_n = \hbar\omega(n + \frac{1}{2})$ where $n = 0, 1, 2, \ldots$ Let n_F denote the state corresponding to Fermi energy.

Then $n_F = N/2$, $E_F = \hbar\omega(\frac{N}{2} + \frac{1}{2}) \simeq \frac{\hbar\omega N}{2}$

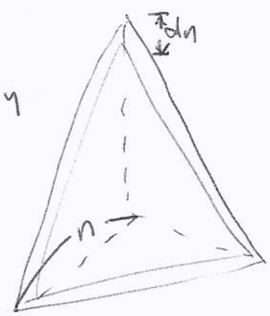
Ground state energy: $2E_0 + 2E_1 + 2E_2 + \cdots + 2E_F$

$$= 2(\hbar\omega)\left[\frac{1}{2} + \frac{3}{2} + \cdots + (n_F + \tfrac{1}{2})\right] = \hbar\omega(n_F+1)^2 \simeq \hbar\omega\left(\frac{N}{2}\right)^2$$

B. The volume in n-space: $\frac{1}{6}n_F^3$ Electrons per unit volume: 2



Thus, $n_F = (3N)^{1/3}$, $E_F = \hbar\omega(3N)^{1/3}$



$$dV = \left(\frac{\sqrt{3}}{2}n^2\right)\left(\frac{dn}{\sqrt{3}}\right) = \frac{1}{2}n^2 dn$$

$$dE = 2(E_n)\,dV = \hbar\omega(n + \tfrac{3}{2})\,n^2\,dn$$
$$\simeq \hbar\omega n^3 dn$$

$$E_{total} = \int_0^{n_F} \hbar\omega n^3 dn = \frac{\hbar\omega}{4}n_F^4 = \frac{\hbar\omega}{4}(3N)^{4/3}$$

4.

A) $\frac{d}{dt}\langle\psi|\hat{x}\hat{p}|\psi\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{x}\hat{p}]\rangle$ $\qquad \hat{H} = \hat{T} + \hat{V}$

$$= \frac{i}{\hbar}\left(\langle[\hat{H},\hat{x}]\hat{p}\rangle + \langle\hat{x}[\hat{H},\hat{p}]\rangle\right) = \frac{i}{\hbar}\langle[\hat{T},\hat{x}]\hat{p}\rangle + \frac{i}{\hbar}\langle\hat{x}[\hat{V},\hat{p}]\rangle$$

$$[\hat{T},\hat{x}] = \frac{1}{2m}\left(\hat{p}[\hat{p},\hat{x}] + [\hat{p},\hat{x}]\hat{p}\right) = \frac{-i\hbar\hat{p}}{m}$$

$$\frac{i}{\hbar}\langle[\hat{T},\hat{x}]\hat{p}\rangle = \frac{1}{m}\langle\hat{p}^2\rangle = 2\langle\hat{T}\rangle$$

$$\frac{i}{\hbar}\langle\hat{x}[\hat{V},\hat{p}]\rangle = \int\psi^* x\left(V\frac{d\psi}{dx} - \frac{d}{dx}(V\psi)\right)dx = -\int\psi^* x\frac{dV}{dx}\psi\, dx = -\langle x\frac{dV}{dx}\rangle$$

For stationary states, $\frac{d}{dt}\langle\hat{x}\hat{p}\rangle = 0$ $\quad\Rightarrow\quad 2\langle T\rangle = \langle x\frac{dV}{dx}\rangle$ ☑

B)

$$2\langle T_x\rangle = \langle x\frac{dV}{dx}\rangle \qquad 2\langle T_y\rangle = \langle y\frac{dV}{dy}\rangle \qquad 2\langle T_z\rangle = \langle z\frac{dV}{dz}\rangle$$

$\frac{P_x^2}{2m}$

Adding these 3 equations yields : $\qquad T = \frac{P_x^2 + P_y^2 + P_z^2}{2m} = T_x + T_y + T_z$

$$2\langle T\rangle = \langle(x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz})V\rangle = \langle\vec{r}\cdot\vec{\nabla}V\rangle$$

5.

A) Given $\hat{a}_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} \mp ip)$ so $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)$

$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-)$

$\hat{a}_+ |n\rangle = (n+1)\,|n+1\rangle \qquad \langle m|n\rangle = \delta_{mn}$

$\hat{a}_- |n\rangle = n\,|n-1\rangle \qquad \langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle n|\,\hat{a}_+ + \hat{a}_-\,|n\rangle = 0$

$$\langle \hat{x}^2\rangle = \frac{\hbar}{2m\omega}\langle n|\,a_+a_- + a_-a_+\,|n\rangle = \frac{\hbar}{2m\omega}(2n+1)$$

B) $\langle \hat{p}\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\langle n|\,\hat{a}_+ - \hat{a}_-\,|n\rangle = 0$

$$\langle \hat{p}^2\rangle = -\frac{\hbar m\omega}{2}\langle n|\,\cancel{\hat{a}_+^2}^{\,0} - \hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_- - \cancel{\hat{a}_-^2}\,|n\rangle = \frac{\hbar m\omega}{2}(2n+1)$$

C) $\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{\hbar(2n+1)}{2m\omega}}$

$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{\hbar(2n+1)m\omega}{2}}$

$\sigma_x \cdot \sigma_p = \hbar(n+\tfrac{1}{2}) \geq \frac{\hbar}{2}$

D) $\langle T\rangle = \frac{\langle p^2\rangle}{2m} = \frac{\hbar\omega}{2}(n+\tfrac{1}{2}) \qquad \langle V\rangle = \frac{m\omega^2}{2}\langle x^2\rangle = \frac{\hbar\omega}{2}(n+\tfrac{1}{2})$

$$\langle x\frac{dV}{dx}\rangle = \langle x\,\frac{m\omega^2}{2}\,2x\rangle = 2\langle V\rangle = 2\langle T\rangle$$

6. $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

A) $S_i |i\rangle = 0 \quad |x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad |y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad |z\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\langle i | j \rangle = \delta_{ij} \quad i, j = x, y, z$

B) $\hat{Q} = a|x\rangle\langle x| + b|y\rangle\langle y| + c|z\rangle\langle z|$

By construction of $\hat{Q}$, $\hat{Q}|x\rangle = a|x\rangle \quad \hat{Q}|y\rangle = b|y\rangle \quad \hat{Q}|z\rangle = c|z\rangle$

eigenvalue eigenvector

C) $\hat{Q} = \frac{a}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} + \frac{b}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \frac{c}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a+b & 0 & -a+b \\ 0 & c & 0 \\ -a+b & 0 & a+b \end{pmatrix}$

$\langle 1 | \hat{Q} | -1 \rangle = \frac{-a+b}{2}$ but $\langle 1 | \vec{B}\cdot\vec{S} | -1 \rangle = \sum_{i=x}^{z} B_i \cancelto{0}{\langle 1 | S_i | -1 \rangle} = 0$

$\hat{Q} = \vec{B}\cdot\vec{S}$ only if $\frac{-a+b}{2} = 0$ but $a \neq b$ stated in (B).

D) To separate $|x\rangle$, $|y\rangle$, $|z\rangle$ states in a $\vec{B}$ field, we need to make these 3 states eigenstates of energy with 3 distinct eigenvalues, whose operator takes the general form of $\hat{Q}$.

However, in a Stern-Gerlach experiment, $\hat{H} \propto \vec{B}\cdot\vec{S}$ and $\vec{B}\cdot\vec{S} \neq \hat{Q}$ for any $\vec{B}$.

Thus, $|x\rangle$, $|y\rangle$, $|z\rangle$ states can never be the eigenstates of energy in a Stern-Gerlach experiment.