Physics 137A

Lecture 1 ⋄ Spring 2014
University of California at Berkeley

Final Exam

May 12, 2014, 7-10pm, 4 LeConte
6 problems ⋄ 180minutes ⋄ 100points

Problem 1 ⋄ THREE-DIMENSIONAL VECTOR SPACE *10points*

Consider a three-dimensional vector space spanned by an orthonormal basis $\{|1\rangle, |2\rangle, |3\rangle\}$. Kets $|\alpha\rangle$ and $|\beta\rangle$ are given by:

$$|\alpha\rangle = i\,|1\rangle - 5\,|2\rangle - i\,|3\rangle\,, \quad |\beta\rangle = i\,|1\rangle + 3\,|3\rangle\,.$$

⋄ A ⋄ Construct bras $\langle\alpha|$ and $\langle\beta|$ in terms of the dual basis vectors $\{\langle 1|, \langle 2|, \langle 3|\}$.
⋄ B ⋄ Find $\langle\alpha\,|\,\beta\rangle$ and $\langle\beta\,|\,\alpha\rangle$.
⋄ C ⋄ Find all matrix elements of the operator $\hat{A} = |\beta\rangle\,\langle\alpha|$, in this basis, and write this operator as a matrix. Is it Hermitian?

Problem 2 ⋄ FOUR PARTICLES IN A SQUARE WELL *20points*

Consider a set of four noninteracting identical particles of mass m confined in a one-dimensional infinitely high square well of length L.

⋄ A ⋄ What are the single particle energy levels? What are the corresponding single particle wave functions? Name the wave functions $\phi_1(x)$, $\phi_2(x)$, and so on with the corresponding energies ϵ_1, ϵ_2, etc.
⋄ B ⋄ Suppose the particles are spinless bosons. What is the energy and (properly normalized) wave function of the grounds state? Of the first excited state? Of the second excited state? Express these three states $\psi_n(x_1, x_2, x_3, x_4)$ and corresponding energies E_n in terms of your answers from part A: ϕ_i's and ϵ_i's.
⋄ C ⋄ If the particles are spin-$\frac{1}{2}$ fermions what is the energy and (properly normalized) wave function of the ground state? The first excited state? The second excited state? Express your answer in terms of single particle wavefunctions and energies from part A. Feel free to introduce convenient notation for single particle spin states and write your answer using a Slater determinant. Note degeneracy of these levels, if any.

Problem 3 ⋄ TRIPLE SPIKE *30points*

Consider the scattering of a particle of mass m with energy $E \gg \frac{m\alpha}{2\hbar^2}$ from a one-dimensional δ-function potentials.

⋄ A ⋄ Find the reflection coefficient from a single δ-function spike at the origin: $V(x) = \alpha\delta(x)$, with $\alpha > 0$. What does the condition $E \gg \frac{m\alpha}{2\hbar^2}$ imply?
⋄ B ⋄ Now two more δ-functions are added to the potential, one to the left and one to the right of the origin:

$$V(x) = \alpha\left[\delta(x+a) + \delta(x) + \delta(x-b)\right], \text{with } \alpha > 0.$$

Find the the relative positions of the potential spikes (a and b) that maximize the reflection coefficient from this triple spike potential.
⋄ C ⋄ How does the reflection coefficient in the arrangement of part B compare to the reflection coefficient from a single δ-function potential?

Problem 4 ⋄ ORBITAL ANGULAR MOMENTUM TWO *15points*

A quantum particle is known to be in an orbital with $l = 2$. You can use the eigenstates of L_z, the z-component of orbital angular momentum, as a basis of this $l = 2$ subspace and denote them $|2\ m_l\rangle$.

⋄ A ⋄ What are allowed values of m_l?
⋄ B ⋄ Find matrix representation of the operators $\hat{L}^2$, $\hat{L}_z$, $\hat{L}_+$, $\hat{L}_-$, $\hat{L}_x$, and $\hat{L}_y$ in this basis.
⋄ C ⋄ Verify explicitly that $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$ in the $l = 2$ subspace.

Problem 5 ◇ ADDITION OF ANGULAR MOMENTUM *15points*

An electron in a hydrogen atom is in an orbital with $l = 2$.

◇ A ◇ What are the possible values of the total angular momentum quantum number j?

◇ B ◇ If the electron is in a state with the lowest j (among those which you found in part A), what are the possible results of a measurement of $\hat{J}_z$, the z-component of the total angular momentum?

◇ C ◇ Suppose that your measurement of $\hat{J}_z$ in part B resulted in $m_j = j$. If you now measure $\hat{L}_z$, the z-component of the orbital part of angular momentum, what are the possible outcomes?

Problem 6 ◇ NONCOMMUTING OPERATORS *10points*

◇ A ◇ Prove that two noncommuting operators cannot have a complete set of common eigenfunctions.

◇ B ◇ Derive the upper limit of the the expectation value of a commutator of two operators, i.e. derive the the generalized uncertainty principle.

You may need to use the Cauchy-Schwarz inequality:

$$\langle f \,|\, f\rangle \, \langle g \,|\, g\rangle \geq |\, \langle f \,|\, g\rangle \,|^2,$$

which holds for any $|f\rangle$ and $|g\rangle$ in a inner product space.

Mathematical Formulas

Trigonometry:

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Gradient operator:

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi}$$

Laplace operator:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\right)$$

Integrals:

$$\int x\sin(ax)\ dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax)$$

$$\int x\cos(ax)\ dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$$

Exponential integrals:

$$\int_0^\infty x^n e^{-x/a}\ dx = n!\ a^{n+1}$$

Gaussian integrals:

$$\int_0^\infty x^{2n} e^{-x^2/a^2}\ dx = \sqrt{\pi}\ \frac{(2n)!}{n!}\ \left(\frac{a}{2}\right)^{2n+1}$$

$$\int_0^\infty x^{2n+1} e^{-x^2/a^2}\ dx = \frac{n!}{2}\ a^{2n+2}$$

Integration by parts:

$$\int_a^b f\frac{dg}{dx}\ dx = -\int_a^b \frac{df}{dx}g\ dx + fg\Big|_a^b$$

Fundamental Equations

Schrödinger equation:

$$i\hbar\frac{\partial\left|\Psi\right\rangle}{\partial t}=\hat{H}\left|\Psi\right\rangle$$

Time-independent Schrödinger equation:

$$\hat{H}\left|\psi\right\rangle=E\left|\psi\right\rangle,\quad\left|\Psi\right\rangle=\left|\psi\right\rangle e^{-iEt/\hbar}$$

Hamiltonian operator:

$$\hat{H}=\frac{\hat{p}^2}{2m}+V=-\frac{\hbar^2}{2m}\nabla^2+V$$

Position and momentum representations:

$$\left\langle x\,|\,p\right\rangle=\tfrac{1}{\sqrt{2\pi\hbar}}exp(\tfrac{ipx}{\hbar}),\quad\psi(x)=\left\langle x\,|\,\psi\right\rangle,\quad\phi(p)=\left\langle p\,|\,\phi\right\rangle,\quad\left\langle x\,|\,\hat{p}\,|\,\psi\right\rangle=-i\hbar\tfrac{d}{dx}\psi(x)$$

Momentum operator:

$$\hat{p}_x=-i\hbar\tfrac{\partial}{\partial x},\quad\hat{p}_y=-i\hbar\tfrac{\partial}{\partial y},\quad\hat{p}_z=-i\hbar\tfrac{\partial}{\partial z}$$

Time dependence of an expectation value:

$$\frac{d\langle\hat{Q}\rangle}{dt}=\frac{i}{\hbar}\left\langle[\hat{H},\hat{Q}]\right\rangle+\left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle$$

Generalized uncertainty principle:

$$\sigma_A\sigma_B\geq\left|\frac{1}{2i}\left\langle[\hat{A},\hat{B}]\right\rangle\right|$$

Canonical commutator:

$$[\hat{x},\hat{p}_x]=i\hbar,\quad[\hat{y},\hat{p}_y]=i\hbar,\quad[\hat{z},\hat{p}_z]=i\hbar$$

Angular momentum:

$$[\hat{L}_x,\hat{L}_y]=i\hbar\hat{L}_z,\quad[\hat{L}_y,\hat{L}_z]=i\hbar\hat{L}_x,\quad[\hat{L}_z,\hat{L}_x]=i\hbar\hat{L}_y$$

Raising and lowering operator for angular momentum:

$$\hat{L}_\pm=\hat{L}_x\pm i\hat{L}_y,\quad[\hat{L}_+,\hat{L}_-]=2\hbar\hat{L}_z,\quad\hat{L}_\pm\left|l,m\right\rangle=\hbar\sqrt{l(l+1)-m(m\pm1)}\left|l,m\pm1\right\rangle$$

Raising and lowering operator for harmonic oscillator:

$$\hat{a}_\pm=\tfrac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x}\mp i\hat{p}),\quad[\hat{a}_-,\hat{a}_+]=1,\quad\hat{a}_+\psi_n=\sqrt{n+1}\psi_{n+1},\quad\hat{a}_-\psi_n=\sqrt{n}\psi_{n-1}$$

Pauli matrices for spin-$\frac{1}{2}$ particle:

$$\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad\sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad\sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

1.

A) $(a|i\rangle + b|j\rangle + c|k\rangle)^\dagger = a^*\langle i| + b^*\langle j| + c^*\langle k|$

$\langle\alpha| = -i\langle 1| - 5\langle 2| + i\langle 3|$

$\langle\beta| = -i\langle 1| + 3\langle 3|$

B) $\langle\alpha|\beta\rangle = (-i)(i) + (-5)(0) + (i)(3)$ using $\langle i|j\rangle = \delta_{ij}$
$= 1 + 3i$

$\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^* = 1 - 3i$

C) $\hat{A} = |\beta\rangle\langle\alpha| = (i|1\rangle + 3|3\rangle)(-i\langle 1| - 5\langle 2| + i\langle 3|)$

$\hat{A} = |1\rangle\langle 1| - 5i|1\rangle\langle 2| - |1\rangle\langle 3| - 3i|3\rangle\langle 1| - 15|3\rangle\langle 2| + 3i|3\rangle\langle 3|$

$\hat{A} = \begin{pmatrix} 1 & -5i & -1 \\ 0 & 0 & 0 \\ -3i & -15 & 3i \end{pmatrix}$ $\quad \hat{A}^\dagger = (\hat{A}^T)^* \neq \hat{A}$ not Hermitian.

PROBLEM 2

A. $\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$ $\qquad E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2}$

B. Ground state:

$$\Psi_0(x_1 x_2 x_3 x_4) = \phi_1(x_1)\phi_1(x_2)\phi_1(x_3)\phi_1(x_4) \qquad E_0 = \frac{\hbar^2\pi^2}{2mL^2}(4)$$

1st excited state:

$$\Psi_1(x_1 x_2 x_3 x_4) = \frac{1}{\sqrt{4}}\left\{\begin{array}{l} \phi_1(x_1)\phi_1(x_2)\phi_1(x_3)\phi_2(x_4) \\ + \phi_1(x_1)\phi_1(x_2)\phi_2(x_3)\phi_1(x_4) \\ + \phi_1(x_1)\phi_2(x_2)\phi_1(x_3)\phi_1(x_4) \\ + \phi_2(x_1)\phi_1(x_2)\phi_1(x_3)\phi_1(x_4) \end{array}\right\} \qquad E_1 = \frac{\hbar^2\pi^2}{2mL^2}(7)$$

2nd excited state:

$$\Psi_2(x_1 x_2 x_3 x_4) = \frac{1}{\sqrt{6}}\left\{\begin{array}{l} \phi_1(x_1)\phi_1(x_2)\phi_2(x_3)\phi_2(x_4) \\ + \phi_1(x_1)\phi_2(x_2)\phi_2(x_3)\phi_1(x_4) \\ + \phi_2(x_1)\phi_2(x_2)\phi_1(x_3)\phi_1(x_4) \\ + \phi_1(x_1)\phi_2(x_2)\phi_1(x_3)\phi_2(x_4) \\ + \phi_2(x_1)\phi_1(x_2)\phi_2(x_3)\phi_1(x_4) \\ + \phi_2(x_1)\phi_1(x_2)\phi_1(x_3)\phi_2(x_4) \end{array}\right\} \qquad E_2 = \frac{\hbar^2\pi^2}{2mL^2}(10)$$

C.

Ground State

$$\Psi_0(x_1 x_2 x_3 x_4) = \frac{1}{\sqrt{4!}}\begin{vmatrix} \phi_1(x_1)|\uparrow\rangle & \phi_1(x_1)|\downarrow\rangle & \phi_2(x_1)|\uparrow\rangle & \phi_2(x_1)|\downarrow\rangle \\ \phi_1(x_2)|\uparrow\rangle & \phi_1(x_2)|\downarrow\rangle & \phi_2(x_2)|\uparrow\rangle & \phi_2(x_2)|\downarrow\rangle \\ \phi_1(x_3)|\uparrow\rangle & \phi_1(x_3)|\downarrow\rangle & \phi_2(x_3)|\uparrow\rangle & \phi_2(x_3)|\downarrow\rangle \\ \phi_1(x_4)|\uparrow\rangle & \phi_1(x_4)|\downarrow\rangle & \phi_2(x_4)|\uparrow\rangle & \phi_2(x_4)|\downarrow\rangle \end{vmatrix}$$

$$E_0 = \frac{\hbar^2\pi^2}{2mL^2}(10)$$

1st excited state

$$\psi_1(x_1x_2x_3x_4) = \frac{1}{\sqrt{4!}} \times \begin{vmatrix} \phi_1(x_1)|\uparrow\rangle & \phi_1(x_1)|\downarrow\rangle & \phi_2(x_1)|\chi_1\rangle & \phi_3(x_1)|\chi_2\rangle \\ \phi_1(x_2)|\uparrow\rangle & \phi_1(x_2)|\downarrow\rangle & \phi_2(x_2)|\chi_1\rangle & \phi_3(x_2)|\chi_2\rangle \\ \phi_1(x_3)|\uparrow\rangle & \phi_1(x_3)|\downarrow\rangle & \phi_2(x_3)|\chi_1\rangle & \phi_3(x_3)|\chi_2\rangle \\ \phi_1(x_4)|\uparrow\rangle & \phi_1(x_4)|\downarrow\rangle & \phi_2(x_4)|\chi_1\rangle & \phi_3(x_4)|\chi_2\rangle \end{vmatrix}$$

$$E_1 = \frac{\hbar^2\pi^2}{2mL^2} \times (15)$$

There is degeneracy because $|\chi_1\rangle$ and $|\chi_2\rangle$ can be any arbitrary linear combination of spin up & down states.

2nd excited state

$$\psi_2(x_1x_2x_3x_4) = \frac{1}{\sqrt{4!}} \times \begin{vmatrix} \phi_1(x_1)|\uparrow\rangle & \phi_2(x_1)|\uparrow\rangle & \phi_2(x_1)|\downarrow\rangle & \phi_3(x_1)|\chi\rangle \\ \phi_1(x_2)|\uparrow\rangle & \vdots & \vdots & \vdots \\ \phi_1(x_3)|\uparrow\rangle & & & \\ \phi_1(x_4)|\uparrow\rangle & & & \end{vmatrix}$$

$$E_2 = \frac{\hbar^2\pi^2}{2mL^2}(1+4+4+9) = \frac{\hbar^2\pi^2}{2mL^2}(18)$$

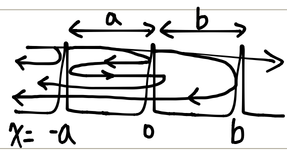
Because $|\chi\rangle$ can be any spin state, this is also degenerate.

3.

A) $R = \dfrac{1}{1+\frac{2\hbar^2 E}{m\alpha^2}}$ Refer to Griffiths for derivation of Eqn. 2.141

$E \gg \dfrac{m\alpha^2}{2\hbar^2}$ $\quad \dfrac{2\hbar^2 E}{m\alpha^2} \gg 1 \Rightarrow R \ll 1.$

B) Actual wave reflection coefficient includes secondary and further refletions as illustrated below.



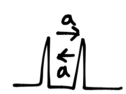
However, $R \ll 1$, these further refletions can be ignored.

Then the net reflected wave can be obtained by the interference of three reflected waves.

Let $\Psi_{in} = Ae^{i(kx-\omega t)}$ and $\Psi_{re} = Be^{i(-kx-\omega t)}$ for one single well.

$$\Psi_{RE} = Be^{i(-kx-\omega t)} + Be^{i(-kx-\omega t - k2a)} + Be^{i(-kx-\omega t - k2(a+b))}$$

phase phase



Please note the addition phases of 2^{nd} & 3^{rd} reflected waves are from the additional distances travelled. $\Psi_{RE} = \Psi_{re}\left(1 + e^{-i2ka} + e^{-i2k(a+b)}\right)$

$$R_{new} = \left|\frac{\Psi_{RE}}{\Psi_{in}}\right|^2 = R_{old}\left|1 + e^{-i2ka} + e^{-i2k(a+b)}\right|^2$$

This is maximized when $2ka = 2\pi n$ $\quad 2k(a+b) = 2\pi m$ $\quad n, m \in \mathbb{Z}$ (constructive interference)

$a = \frac{n\pi}{k}$ $\quad b = \frac{m\pi}{k} - \frac{n\pi}{k} = (m-n)\frac{\pi}{k} = \frac{q\pi}{k}$ $\quad q \in \mathbb{Z}.$

C) When $a = \frac{n\pi}{k}$ $\quad b = \frac{q\pi}{k}$, $\quad R_{new} = R_{old}|3|^3 = 9R_{old}.$

PROBLEM 4

A. $m_\ell = -2, -1, 0, +1, +2$

B.

$$L^2 = 6\hbar \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad L_z = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

$$L_+ = \hbar \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad L_- = \hbar \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

$$L_x = \frac{1}{2}(L_+ + L_-) = \frac{\hbar}{2} \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix} \qquad L_y = \frac{1}{2i}(L_+ - L_-) = \frac{\hbar}{2i} \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & -\sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & -\sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

C. $L_x L_y - L_y L_x$

$$= \left(\frac{\hbar^2}{4i}\right) \begin{bmatrix} -4 & 0 & 2\sqrt{6} & 0 & 0 \\ 0 & -2 & 0 & 6 & 0 \\ -2\sqrt{6} & 0 & 0 & 0 & 2\sqrt{6} \\ 0 & -6 & 0 & 2 & 0 \\ 0 & 0 & -2\sqrt{6} & 0 & 4 \end{bmatrix} - \left(\frac{\hbar^2}{4i}\right) \begin{bmatrix} 4 & 0 & 2\sqrt{6} & 0 & 0 \\ 0 & 2 & 0 & 6 & 0 \\ -2\sqrt{6} & 0 & 0 & 0 & 2\sqrt{6} \\ 0 & -6 & 0 & -2 & 0 \\ 0 & 0 & -2\sqrt{6} & 0 & -4 \end{bmatrix}$$

$$= i\hbar L_z$$

PROBLEM 5

A. The total angular momentum can be any value between ~~$|S_1 - S_2|, \ldots, (S_1 +$~~ $|S_1 - S_2|, \ldots, (S_1 + S_2)$ in integer steps.

$j = 3/2$ or $5/2$

B. If $j = 3/2$, then m_j can take any value between $-j$ and j in integer steps. $m_j = +3/2, +1/2, -1/2, -3/2$

C. $|3/2\ 3/2\rangle$ state is a linear combination of $|2\ 1\rangle|1/2\ 1/2\rangle$ state and $|2\ 2\rangle|1/2\ -1/2\rangle$ state. Thus, measurement of L_z would yield either $\hbar$ or $2\hbar$.

6.

A) We will prove it by its contrapositive statement:
If $\hat{P}$ and $\hat{Q}$ have common eigenfunctions, they have to commute.

Let ψ_n be the common eigenfunctions, i.e. $\hat{P}\psi_n = P_n\psi_n \quad \hat{Q}\psi_n = Q_n\psi_n$

Any f can be expressed as $f = \sum_n c_n \psi_n$.

$$[\hat{P}, \hat{Q}] f = \sum_n c_n (\hat{P}\hat{Q} - \hat{Q}\hat{P}) \psi_n = \sum_n c_n (\cancelto{0}{P_n Q_n - Q_n P_n}) \psi_n = 0$$

B) Please refer to section 3.5 on p.110 of Griffiths.