# Physics 137A

Spring 2014
University of California at Berkeley

# Midterm 2

April 14, 2014, 7-9pm, 1 LeConte
*120minutes ⋄ 50points*

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*Problem 1* ⋄ Sequential Measurements (Modified Griffiths Problem 3.27) *7points*

An operator $\hat{A}$, corresponding to an observable $\alpha$, has two normalized eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$, with eigenvalues $a_1$ and $a_2$. An operator $\hat{B}$, corresponding to another observable $\beta$, has two normalized eigenstates $|\phi_1\rangle$ and $|\phi_2\rangle$, with eigenvalues $b_1$ and $b_2$. The eigenstate are related by:

$$|\psi_1\rangle = \frac{3}{5}|\phi_1\rangle + \frac{4}{5}|\phi_2\rangle\,, \quad |\psi_2\rangle = \frac{4}{5}|\phi_1\rangle - \frac{3}{5}|\phi_2\rangle\,.$$

⋄ A ⋄ Observable $\beta$ is measured and value $b_1$ is obtained. What is the state of the system after this measurement?

⋄ B ⋄ If observable $\alpha$ is subsequently measured, and then $\beta$ is measured again, what is the probability that the value $b_1$ is obtained the second time $\beta$ is measured?

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*Problem 2* ⋄ Harmonic Oscillator VS Hydrogen Atom Ground State *12points*

⋄ A ⋄ Show that $\psi_0(r) \propto e^{-ar^2}$ is a solution of the Schrödinger equation for the harmonic oscillator, provided that $a = \frac{m\omega}{2\hbar}$. Find the corresponding ground state energy $E_0$. Normalize the wave function.

⋄ B ⋄ Show that $\psi_1(r) \propto e^{-br}$ is a solution of the Schrödinger equation for the hydrogen atom, provided that $b = \frac{mke^2}{\hbar^2} = \frac{1}{a_0}$, where $a_0$ is the Bohr radius. Find the corresponding ground state energy $E_1$. Normalize the wave function.

⋄ C ⋄ If we move along, say, the $x$-axis, there is a kink (a discontinuity in slope) in the hydrogen atom ground state $\psi_1(r)$ on passing through the origin. Why (physically) is there one here but not for the harmonic oscillator $\psi_0(r)$?

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*Problem 3* ⋄ Hydrogen Atom Ground States *20points*

A hydrogen atom is in its ground state just like in *Problem 2*B.

⋄ A ⋄ If space is divided into identical infinitesimal cubes, in which cube is the electron most likely to be found?

⋄ B ⋄ If instead space is divided into shells of infinitesimal thickness, like the layers of an onion, centered on the proton, what is the radius of the shell in which the electron is most likely to be found?

⋄ C ⋄ Calculate the mean radius of hydrogen atom $\langle r\rangle$?

⋄ D ⋄ Calculate the mean value of the potential energy $V(r)$.

⋄ E ⋄ What is the mean value of the kinetic energy $T$? (No calculation needed here!)

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*Problem 4* ⋄ Measuring Electron's Spin (Griffiths Problem 4.49) *11points*

An electron at rest is in the spin state given by the spinor

$$|\chi\rangle = N\begin{pmatrix} 1-2i \\ 2 \end{pmatrix}$$

in the standard basis of eigenstates of $\hat{S}_z$ with spin up $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and spin down $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

⋄ A ⋄ Determine the constant $N$ by normalizing $|\chi\rangle$.

⋄ B ⋄ If you measured $S_z$ on this electron, what values could you get, and what is the probability of each? What is the expectation value of $S_z$?

⋄ C ⋄ If you measured $S_x$ on this electron, what values could you get, and what is the probability of each? What is the expectation value of $S_x$?

⋄ D ⋄ If you measured $S_y$ on this electron, what values could you get, and what is the probability of each? What is the expectation value of $S_y$?

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# Mathematical Formulas

Trigonometry:

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Integrals:

$$\int x \sin(ax)\ dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax)$$

$$\int x \cos(ax)\ dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$$

Exponential integrals:

$$\int_0^\infty x^n e^{-x/a}\ dx = n!\ a^{n+1}$$

Gaussian integrals:

$$\int_0^\infty x^{2n} e^{-x^2/a^2}\ dx = \sqrt{\pi}\ \frac{(2n)!}{n!}\ \left(\frac{a}{2}\right)^{2n+1}$$

$$\int_0^\infty x^{2n+1} e^{-x^2/a^2}\ dx = \frac{n!}{2}\ a^{2n+2}$$

Integration by parts:

$$\int_a^b f\frac{dg}{dx}\ dx = -\int_a^b \frac{df}{dx}g\ dx + fg\Big|_a^b$$

# Fundamental Equations

Schrödinger equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$$

Time-independent Schrödinger equation:

$$\hat{H}\psi = E\psi, \quad \Psi = \psi e^{-iEt/\hbar}$$

Hamiltonian operator:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V = -\frac{\hbar^2}{2m}\nabla^2 + V = -\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\right)\right] + V$$

Momentum operator:

$$\hat{p}_x = -i\hbar\tfrac{\partial}{\partial x}, \quad \hat{p}_y = -i\hbar\tfrac{\partial}{\partial y}, \quad \hat{p}_z = -i\hbar\tfrac{\partial}{\partial z}$$

Time dependence of an expectation value:

$$\frac{d\langle\hat{Q}\rangle}{dt} = \frac{i}{\hbar}\left\langle[\hat{H},\hat{Q}]\right\rangle + \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle$$

Generalized uncertainty principle:

$$\sigma_A\sigma_B \geq \left|\frac{1}{2i}\left\langle[\hat{A},\hat{B}]\right\rangle\right|$$

Canonical commutator:

$$[\hat{x},\hat{p}_x] = i\hbar, \quad [\hat{y},\hat{p}_y] = i\hbar, \quad [\hat{z},\hat{p}_z] = i\hbar$$

Angular momentum:

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z, \quad [\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x, \quad [\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$$

Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1.

A) After the measurement, the wave function collaspes into the eigenstate of the measured value. Hence, $|\phi_1\rangle$.

B) Solving for $|\phi_1\rangle$, it gives $|\phi_1\rangle = \frac{3}{5}|\psi_1\rangle + \frac{4}{5}|\psi_2\rangle$

Measuring $\alpha$

$$a_1 \quad P_1 = \frac{9}{25} \rightarrow |\psi_1\rangle \xrightarrow{\text{to get } b_1} P_1' = \frac{9}{25}$$

$$a_2 \quad P_2 = \frac{16}{25} \rightarrow |\psi_2\rangle \xrightarrow{\phantom{\text{to get } b_1}} P_2' = \frac{16}{25}$$

$$P_{tot} = \left(\frac{9}{25}\right)^2 + \left(\frac{16}{25}\right)^2 = \frac{337}{625}$$

2. A) $\psi_0 = Ae^{-ar^2}$ $\quad H\psi = \frac{-\hbar^2}{2m}\nabla^2\psi + \frac{1}{2}m\omega^2 r^2\psi = E\psi$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \cdots \text{ (angular derivatives)}$$

$$H\psi = -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(-2ar^3 A\, e^{-ar^2}\right) + \frac{1}{2}m\omega^2 r^2 A\, e^{-2ar^2}$$

$$-\frac{\hbar^2}{2m}\left(-6aA\,e^{-ar^2} + \cancel{4a^2r^2A\,e^{-ar^2}}\right) + \cancel{\frac{1}{2}m\omega^2r^2A\,e^{-ar^2}} \propto \psi$$

since $-\frac{\hbar^2}{2m}4a^2r^2 = \frac{-1}{2}m\omega^2r^2$ so $\psi_0$ solves $H\psi = E\psi$ ☑

$$-\frac{\hbar^2}{2m}\left(-6a\cancel{A}\cancel{e^{-ar^2}}\right) = E\cancel{A}\cancel{e^{-ar^2}} \qquad E = \frac{3\hbar^2}{m}a = \boxed{\frac{3}{2}\hbar\omega} \text{ as expected.}$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} A^2 e^{-2ar^2} r^2 \sin\theta\, dr\, d\theta\, d\phi = A^2\cdot 4\pi\cdot\sqrt{\pi}\,\frac{2!}{1!}\left(\frac{1}{2\sqrt{2a}}\right)^3 = 1 \qquad A = \left(\frac{2a}{\pi}\right)^{3/4}$$

$$\psi_0 = \left(\frac{2a}{\pi}\right)^{3/4} e^{-ar^2}$$

☑

B) $\psi_0 = Ae^{-br}$ $\quad H\psi = \frac{-\hbar^2}{2m}\nabla^2\psi + \frac{-ke^2}{r}\psi = E\psi$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \cdots \text{ (angular derivatives)}$$

$$H\psi = -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(-br^2 A\,e^{-br}\right) + \frac{-ke^2}{r}A\,e^{-br}$$

$$-\frac{\hbar^2}{2m}\left(b^2A\,e^{-br} - \cancel{2b\frac{1}{r}A\,e^{-br}}\right) + \cancel{\frac{-ke^2}{r}A\,e^{-br}} \propto \psi$$

since $-\frac{\hbar^2}{2m}\left(-\frac{2b}{r}\right) = \frac{ke^2}{r}$ so $\psi_0$ solves $H\psi = E\psi$ ☑

$$-\frac{\hbar^2}{2m}\left(b^2\cancel{A}\cancel{e^{-br}}\right) = E\cancel{A}\cancel{e^{-br}} \qquad E = \boxed{-\frac{m}{2}\left(\frac{ke^2}{\hbar}\right)^2}$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} A^2 e^{-2br} r^2\sin\theta\, dr\, d\theta\, d\phi = A^2\cdot 4\pi\cdot 2!\left(\frac{1}{2b}\right)^3 = 1 \qquad A = \left(\frac{b^3}{\pi}\right)^{1/2}$$

$$\psi_0 = \left(\frac{b^3}{\pi}\right)^{1/2} e^{-br}$$ ☑

C) As seen in delta potential $\delta(x)$, a singularity in $V$ causes a kink in $\psi$. Thus, this is because H atom potential is singular at the origin ($\sim\frac{1}{r}$) while H.O. potential is continuous and smooth at the origin ($\sim r^2$).

# PROBLEM 3

A. $\psi_1(r) = \sqrt{\frac{b^3}{\pi}}\, e^{-br}$ or $\frac{1}{\sqrt{\pi a^3}}\, e^{-r/a}$

Probability the particle is in the infinitesmal square : $|\psi_1(r)|^2\, dx\, dy\, dz$

$= \frac{1}{\pi a^3} e^{-2r/a}\, dx\, dy\, dz$. This is maximum at the origin where $r=0$.

B. Probability the particle is in the spherical shell : $|\psi_1(r)|^2\, 4\pi r^2\, dr$

$= \frac{4}{a^3} r^2 e^{-2r/a}\, dr = P(r)\, dr$ where $P(r) = \frac{4}{a^3} r^2 e^{-2r/a}$

max when $\frac{dP}{dr} = 0$. $\frac{dP}{dr} = \frac{4}{a^3}\left[2r e^{-2r/a} - 2b r^2 e^{-2br}\right] \Rightarrow 0$ when $\underline{r = a_0}$

C. $\langle r \rangle = \int_0^\infty |\psi_1|^2 \cdot r \cdot 4\pi r^2\, dr = \frac{4}{a^3}\int_0^\infty r^3 e^{-2br}\, dr = \frac{4}{a^3} \cdot 3! \cdot \left(\frac{a}{2}\right)^4$

$= \underline{\frac{3}{2} a_0}$

D. $\langle V(r) \rangle = \int_0^\infty \psi_1^* V(r) \psi_1\, 4\pi r^2\, dr = \int_0^\infty \left(\frac{1}{\pi a^3} e^{-2r/a}\right)\left(\frac{-ke^2}{r}\right) 4\pi r^2\, dr$

$= -\frac{4ke^2}{a_0^3}\int_0^\infty r e^{-2r/a}\, dr = \frac{-ke^2}{a_0}$

E. Ehrenfest's Thm: KE + PE = Etotal

$\langle V \rangle + \langle T \rangle = E_0 \rightarrow \langle T \rangle = E_0 - \langle V \rangle = -\frac{1}{2}\cdot\frac{ke^2}{a_0} + \frac{ke^2}{a_0} = \underline{\frac{1}{2}\cdot\frac{ke^2}{a_0}}$

$\left(\text{or } -13.6\text{eV} + \frac{ke^2}{a_0}\right)$

= +13.6 eV

## PROBLEM 4

A. $|x\rangle = N\begin{pmatrix} 1-2i \\ 2 \end{pmatrix}$, $\langle x|x\rangle = N^2 (1+2i \quad 2)\begin{pmatrix} 1-2i \\ 2 \end{pmatrix} = 9N^2$

$N = \frac{1}{3}$ $(\langle x|x\rangle = 1)$

B. $|z+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|z-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $|x\rangle = \left(\frac{1-2i}{3}\right)|z+\rangle + \left(\frac{2}{3}\right)|z-\rangle$

Prob of measuring $+\hbar/2$ : $\left|\frac{1-2i}{3}\right|^2 = \underline{\frac{5}{9}}$

prob of measuring $-\hbar/2$ : $\left|\frac{2}{3}\right|^2 = \underline{\frac{4}{9}}$

Expectation value : $\left(\frac{\hbar}{2}\right)\left(\frac{5}{9}\right) + \left(\frac{-\hbar}{2}\right)\left(\frac{4}{9}\right) = \underline{\frac{\hbar}{18}}$

C. Finding eigenspinors for $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

eigenvalue 1: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \to \alpha = \beta$ $\quad |x+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

eigenvalue $-1$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \to \alpha = -\beta$ $\quad |x-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

prob of measuring $+\hbar/2$ : $|\langle x+|x\rangle|^2 = \left|\frac{1}{\sqrt{2}}(1 \quad 1)\frac{1}{3}\begin{pmatrix} 1-2i \\ 2 \end{pmatrix}\right|^2 = \underline{\frac{13}{18}}$

prob of measuring $-\hbar/2$ : $|\langle x-|x\rangle|^2 = \left|\frac{1}{\sqrt{2}}(1 \quad -1)\frac{1}{3}\begin{pmatrix} 1-2i \\ 2 \end{pmatrix}\right|^2 = \underline{\frac{5}{18}}$

$\langle S_x\rangle = \left(\frac{\hbar}{2}\right)\left(\frac{13}{18}\right) + \left(\frac{-\hbar}{2}\right)\left(\frac{5}{18}\right) = \underline{\frac{2}{9}\hbar}$

D. Finding eigenspinors for $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

eigenvalue 1 : $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, $\begin{pmatrix} -i\beta \\ i\alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \to \begin{matrix} \alpha = 1 \\ \beta = i \end{matrix}$ $\quad \underline{|y+\rangle = \begin{pmatrix} 1 \\ i \end{pmatrix}\frac{1}{\sqrt{2}}}$

eigenvalue $-1$ : $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, $\begin{pmatrix} -i\beta \\ i\alpha \end{pmatrix} = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix} \to \begin{matrix} \alpha = 1 \\ \beta = -i \end{matrix}$ $\quad \underline{|y-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}}$

prob of measuring $+\hbar/2$ : $|\langle y+|x\rangle|^2 = \left|\frac{1}{\sqrt{2}}(1 \quad -i)\frac{1}{3}\begin{pmatrix} 1-2i \\ 2 \end{pmatrix}\right|^2 = \underline{\frac{17}{18}}$

prob of measuring $-\hbar/2$ : $|\langle y-|x\rangle|^2 = \left|\frac{1}{\sqrt{2}}(1 \quad i)\frac{1}{3}\begin{pmatrix} 1-2i \\ 2 \end{pmatrix}\right|^2 = \underline{\frac{1}{18}}$

$\langle S_y\rangle = \left(\frac{\hbar}{2}\right)\left(\frac{17}{18}\right) + \left(\frac{-\hbar}{2}\right)\left(\frac{1}{18}\right) = \underline{\frac{4}{9}\hbar}$