MATH 104 FINAL SOLUTIONS

1. (2 points each) Mark each of the following as True or False. No justification is required.

a) An unbounded sequence can have no Cauchy subsequence. **False**

b) An infinite union of Dedekind cuts is a Dedekind cut. **False**

c) If $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$, there is a sequence of polynomials whose uniform limit on $[a, b]$ is f. **True**

d) If $f_n \to f$ uniformly on S, then $f_n' \to f'$ uniformly on S. **False**

e) If f is differentiable on $[a, b]$ then it is integrable on $[a, b]$. **True**

2. Given a sequence $\{x_n\}$, define a sequence $\{y_n\}$ by setting

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

a) (6 points) If $x_n \to a \in \mathbb{R}$, show that $y_n \to a \in \mathbb{R}$.

For any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|x_n - a| < \epsilon$ for all $n > N$. Also, since x_n converges, it is bounded and we have that $|x_n - a| < M$ for all $n \in \mathbb{N}$, for some $M \in \mathbb{R}$. Let $\epsilon' = (M + 1)\epsilon$. Note that for $n > \max(N, N/\epsilon)$ we have

$$\begin{aligned} |y_n - a| &= \left| \frac{(x_1 - a) + (x_2 - a) + \cdots + (x_n - a)}{n} \right| \\ &\leq \frac{|x_1 - a| + \cdots + |x_n - a|}{n} \\ &\leq \frac{MN}{n} + \frac{\epsilon(n - N)}{n} \\ &< (M + 1)\epsilon = \epsilon' \end{aligned}$$

Hence we have that $y_n \to a$.

b) (4 points) Give an example of a divergent sequence $\{x_n\}$ (i.e. with no limit in $\mathbb{R}$) for which $\{y_n\}$ as defined above converges.

Consider the sequence $\{x_n\} = \{(-1)^n\}$. This is a divergent sequence, but $\{y_n\} = \{-1, 0, -\frac{1}{3}, 0, -\frac{1}{5}, 0, \dots\}$ converges to 0.

3. a) (8 points) Find an example or prove that the following does not exist: a monotone sequence that has no limit in $\mathbb{R}$ but has a subsequence converging to a real number.

Such a thing does not exist. Any bounded monotone sequence has a limit in $\mathbb{R}$, so a monotone sequence $\{s_n\}$ that has no limit in $\mathbb{R}$ is unbounded. Suppose $\{s_n\}$ is increasing (bounded below by

s_1). Since the sequence is unbounded, for any $M \in \mathbb{R}$ there exists some N such that $s_N > M$ and, since s_n is increasing, $s_n > M$ for all $n > N$ as well. Hence $\lim s_n = \infty$ and any subsequence will have the same limit. Similarly, if s_n is decreasing, it is not bounded below, and for any $M \in \mathbb{R}$ there exists some N such that $s_N < M$ and, since s_n is decreasing, $s_n < M$ for all $n > N$ as well. Hence $\lim s_n = -\infty$ and any subsequence will have the same limit.

b) (7 points) Let $x_n = \cos \frac{n\pi}{3}$. Find a convergent subsequence of $\{x_n\}$ and compute $\limsup x_n$.

Consider $\{x_{6n}\} = \{1\}$. This sequence is converges to 1, and, since $\cos x \leq 1$ for all x, we have that this is the largest possible subsequential limit of $\{x_n\}$. Hence $\limsup x_n = 1$.

4) a) (6 points) Consider the series

$$\sum_{n=1}^{\infty} \frac{6^n}{n^n}, \quad \sum_{n=1}^{\infty} \frac{1}{n+1/2}.$$

For each of these, determine whether it converges or diverges and justify your answer.

For the first series, $\limsup \left(\frac{6^n}{n^n}\right)^{1/n} = \lim \frac{6}{n} = 0 < 1$ and so by the root test the series converges. For the second series, note that $\sum_2^\infty \frac{1}{n}$ diverges and has all positive terms, and $\sum_1^\infty \frac{1}{n+1/2} = \sum_2^\infty \frac{1}{n-1/2}$, where $\frac{1}{n-1/2} > \frac{1}{n}$ for all n. Hence by the comparison test this series diverges.

b) (4 points) Let $\{s_n\}$ and $\{t_n\}$ be sequences of positive real numbers. Show that if $s_n/t_n \to 1$ then $\sum s_n$ and $\sum t_n$ either both converge or both diverge.

Since $s_n/t_n \to 1$, there is an $N \in \mathbb{N}$ such that

$$\left|\frac{s_n}{t_n} - 1\right| < \frac{1}{2}$$

and so

$$\frac{1}{2} < \frac{s_n}{t_n} < \frac{3}{2},$$

and

$$\frac{t_n}{2} < s_n < \frac{3t_n}{2}$$

for all $n \geq N$. If $\sum t_n$ converges, then $\sum_N^\infty \frac{3t_n}{2}$ converges and by the above and the comparison test, $\sum_N^\infty s_n$ converges as well. Since $\sum s_n$ and $\sum_N^\infty s_n$ differ by a finite number of terms, one converges if and only if the other does, and so our conclusion holds for $\sum s_n$ as well.

Similarly, if $\sum s_n$ (and hence $\sum 2s_n$) converges, then we have that since $t_n < 2s_n$ for all $n > N$, the series $\sum t_n$ converges by the comparison test. Hence $\sum s_n$ and $\sum t_n$ either both converge or diverge.

5) a) (5 points) Suppose that f_n converges uniformly to f on a set $S \subset \mathbb{R}$, and that g is a bounded

function on S. Prove that the product $g \cdot f_n$ converges uniformly to $g \cdot f$.

If $g(x) = 0$ on S, then $\{gf_n\} = \{0\}$ which converges uniformly to $gf = 0$. Otherwise, let $|g(x)| < M > 0$ for all $x \in S$. For any $\epsilon > 0$, there exists an $N \in \mathbb{R}$ such that $|f_n(x) - f(x)| < \epsilon/M$ for all $x \in S$, for all $n > N$. Then $|g(x)f_n(x) - g(x)f(x)| = |g(x)| \cdot |f_n(x) - f(x)| < \epsilon$ for all $x \in S$, for all $n > N$ and hence gf_n converges uniformly to gf.

b) (5 points) Let $\{f_n\}$ be a sequence of continuous functions on $[a, b]$ that converges uniformly to f on $[a, b]$. Show that if $\{x_n\}$ is a sequence in $[a, b]$ and if $x_n \to x$, then $\lim_{n\to\infty} f_n(x_n) = f(x)$.

Since f_n are continuous on the closed interval $[a, b]$ and converge uniformly to f, we have that f is continuous on $[a, b]$ as well. For any $\epsilon > 0$ there exists an $N \in \mathbb{R}$ such that $|f_n(x_n) - f(x_n)| < \epsilon/2$ for $n > N$. Furthermore, since $x_n \to x$ and $[a, b]$ is closed, we have $x \in [a, b]$. Thus f is continuous at x and there is a $\delta > 0$ such that if $y \in S$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon/2$. Note that there is an $N' \in \mathbb{R}$ such that $|x_n - x| < \delta$ for all $n > N'$, and hence $|f(x_n) - f(x)| < \epsilon/2$ for all $n > N'$. Hence for $n > \max(N, N')$, we have that $|f_n(x_n) - f(x)| < |f_n(x_n) - f(x_n)| + \epsilon/2 < \epsilon$ and so $f_n(x_n) \to f(x)$ as desired.

6) a) (6 points) Prove that $d(x, y) := \min(|x - y|, 1)$ is a metric on $\mathbb{R}$.

First of all, $d(x, y) \in \mathbb{R}$ for all $x, y \in \mathbb{R}$, so d is indeed a function from $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Next, note that $d(x, y) \geq 0$ for all $x, y \in \mathbb{R}$, since both 1 and $|x - y|$ are nonnegative for all $x, y \in \mathbb{R}$. Also, $d(x, y) = 0$ if and only if $d(x, y) = |x - y| = 0$, which is if and only if $x = y$.

Since $|x - y| = |y - x|$, we also have that $d(x, y) = \min(|x - y|, 1) = \min(|y - x|, 1) = d(y, x)$.

Finally, $d(x, y) = \min(|x - y|, 1) \leq \min(|x - z| + |z - y|, 1) \leq \min(|x - z|, 1) + \min(|z - y|, 1) = d(x, z) + d(z, y)$. So all properties of a metric are satisfied.

b) (4 points) Is the set $(-5, 5)$ open with respect to this metric? Prove that your answer is true.

This set is indeed open with respect to the above metric. Let $x \in (-5, 5)$ and let $\delta = \min(|x - 5|, |x + 5|, 1)$. Consider the neighborhood $N(x)$ of radius $\delta/2$ of x with respect to the above metric. Since $\delta/2 < 1$, we have that $d(x, y) = |x - y|$ for $y \in N(x)$, and since $\delta/2 < \min(|x - 5|, |x + 5|)$ we have that $|x - y| < min(|x - 5|, |x + 5|)$. Hence $y \in (-5, 5)$ and $N(x) \subset (-5, 5)$ and x is an interior point in this interval.

7. a) (8 points) Find the Taylor series at 0 for $f(x) = e^x$. Determine its radius of convergence. You can use either the ratio or root test strategy for this.

Since $f^{(n)}(x) = e^x$ for all n, we have that $f^{(n)}(0) = 1$ for all n, and the Taylor series is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Since $\limsup \frac{n!}{(n+1)!} = 0$, we have that the radius of convergence is ∞.

b) (7 points) Prove that the Taylor series in part (a) represents (is equal to) e^x for all $x \in \mathbb{R}$.

To show this, we use Taylor's theorem. In this case,

$$R_n(x) = \frac{e^y}{(n+1)!}x^{n+1}$$

where y is between 0 and x. If $x > 0$ then $e^y < e^x$ and hence

$$0 < R_n(x) < e^x \cdot \frac{x^{n+1}}{(n+1)!} \to 0$$

as $n \to \infty$, and hence $R_n(x) \to 0$ by the squeeze lemma. If $x < 0$, then $x < y < 0$ and so $e^y < 1$ and

$$|R_n(x)| < \frac{|x|^{n+1}}{(n+1)!} \to 0$$

and so again $R_n(x) \to 0$. For $x = 0$, $e^x = 1 = \sum_0^\infty \frac{0^n}{n!}$.

8. (10 points) Let f be a function defined on $\mathbb{R}$ such that

$$|f(x) - f(y)| \leq (x-y)^2$$

for all $x, y \in \mathbb{R}$. Show that f is differentiable on $\mathbb{R}$ and that $f'(x) = 0$ for all $x \in \mathbb{R}$.

Note that for all $x, y \in \mathbb{R}$ we have

$$\left| \frac{f(y) - f(x)}{y - x} \right| \leq |y - x|.$$

Since $|y - x| \to 0$ as $y \to x$, we have by the squeeze lemma that

$$\left| \frac{f(y) - f(x)}{y - x} \right|$$

also tends to 0 as $y \to x$ and so

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = 0.$$

So $f(x)$ is differentiable on $\mathbb{R}$ and its derivative is 0 everywhere.

9) a) (6 points) Let f and g be continuous functions on $[a, b]$ such that $\int_a^b f = \int_a^b g$. Show that there is an $x \in [a, b]$ such that $f(x) = g(x)$.

We have that $\int_a^b (f - g) = 0$. Suppose $f(x) - g(x) \neq 0$ for any $x \in [a, b]$. Since f and g are continuous, so is $f - g$, and hence we have either $f(x) - g(x) < 0$ for all $x \in [a, b]$ or $f(x) - g(x) > 0$ for all $x \in [a, b]$ (otherwise the intermediate value theorem would fail). In the first case, $\int_a^b (f-g) > 0$ since any Riemann sum corresponding to this interval is bounded below by $(b-a) \cdot \min(f(x)-g(x) | x \in [a,b])$ which exists since $f-g$ is continuous on $[a,b]$. In the second case, $\int_a^b (f-g) < 0$ since any Riemann sum corresponding to this interval is bounded below by $(b-a) \cdot \max(f(x)-g(x) | x \in [a,b])$ which exists since $f-g$ is continuous on $[a,b]$. Hence the integral cannot be 0 which is a contradiction, and there must be some $x \in [a,b]$ such that $f(x) = g(x)$.

b) (4 points) Construct an example of functions f, g, both integrable on $[a,b]$, such that $\int_a^b f = \int_a^b g$ but $f(x) \neq g(x)$ for any $x \in [a,b]$.

Let $f(x) = 1$ for $x \in [-1,0]$ and $f(x) = -1$ for $x \in [0,1]$. Let $g(x) = -f(x)$ on $[-1,1]$. Then $\int_{-1}^1 f = \int_{-1}^1 g = 0$ but $f(x) \neq g(x)$ for any $x \in [-1,1]$.