MATH 104 MIDTERM 2 solutions

1. (2 points each) Mark each of the following as True or False. No justification is required.

a) If $\sum a_n$ converges then $\lim a_n = 0$. **True**

b) A real valued function is uniformly continuous on a compact subset of $\mathbb{R}$ if and only if it is continuous on that subset. **True**

c) If $f : \mathbb{R} \to \mathbb{R}$ is defined at $a \in \mathbb{R}$ and $s_n := a + 1/n$, then $\lim f(s_n) = f(a)$. **False**

d) If $f(x)$ and $g(x)$ are uniformly continuous on $\mathbb{R}$, then $f \cdot g$ is uniformly continuous on $\mathbb{R}$. **False**

e) If $\sum a_{2n}$ and $\sum a_{2n+1}$ converge, then so does $\sum a_n$. **True**

2. a) (5 points) Determine whether the following series is absolutely convergent, non-absolutely convergent, or divergent:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$$

This series is absolutely convergent, since $\sum 1/(n^2+1)$ is convergent: we know that $1/n^2 > 0$ for all $n \geq 1$ and $\sum(1/n^2)$ converges. Also, $|1/(n^2+1)| < 1/n^2$ for all $n \geq 1$ and so by the comparison test $\sum 1/(n^2+1)$ converges.

b) (5 points) Suppose a series $\sum a_n$ is convergent and $a_n > 0$ for all n. What, if anything, can you deduce about the convergence or divergence of $\sum \frac{1}{1+a_n}$?

Since $\sum a_n$ converges, we have that $\lim a_n = 0$ and, since $a_n + 1 \neq 0$ for any n, our limit theorems imply that $\lim \frac{1}{1+a_n} = 1/1 = 1 \neq 0$ and hence $\sum \frac{1}{1+a_n}$ diverges.

3. A real-valued function f on a set S is defined to be *Lipschitz continuous* if there exists a $K > 0$ such that for all $x, y \in S$,

$$|f(x) - f(y)| \leq K|x - y|.$$

a) (6 points) Prove that if f is Lipschitz continuous on S, then it is uniformly continuous on S.

Let f be Lipschitz continuous on S with constant K, and let $\varepsilon > 0$. Let $\delta = \varepsilon/(K+1)$. Then if $x, y \in S$ and $|x - y| < \delta$, we have $|f(x) - f(y)| < K|x - y| < K\varepsilon/(K+1) < \varepsilon$ as desired.

b) (4 points) Show that $\sqrt{x}$ is not Lipschitz continuous on $[0, \infty)$.

Suppose $\sqrt{x}$ is Lipschitz continuous on $[0, \infty)$, and so for all $x, y \geq 0$ we have $|\sqrt{x} - \sqrt{y}| \leq K|x - y| = K|\sqrt{x} - \sqrt{y}| \cdot |\sqrt{x} + \sqrt{y}|$. If x or y is nonzero, this means that $1/|\sqrt{x} + \sqrt{y}| \leq K$, but this is not true for any $K > 0$: take $0 < x = y < 1/4K^2$ and we get that $1/|\sqrt{x} + \sqrt{y}| > K$. Hence $\sqrt{x}$ is not Lipschitz continuous on $[0, \infty)$.

4. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function defined on $\mathbb{R}$. Show that the image $f(K) := \{f(k) \mid k \in K\}$ of a compact set K is compact. (*Hint: it may be helpful to show that if U is open then $f^{-1}(U) = \{x \in$*

$\mathbb{R} \mid f(x) \in U\}$ *is open*)

We first prove the statement in the hint. Suppose U is open and let $x \in f^{-1}(U)$. Since $f(x) \in U$, there exists an r such that for all y with $|y - f(x)| < r$ we have $y \in U$. Since f is continuous, there exists a δ such that if $|x' - x| < \delta$ then $|f(x') - f(x)| < r$. Since we have $f(x') \in U$ for all such x', we have $x' \in f^{-1}(U)$ for all $|x - x'| < \delta$ and hence x is an interior point of $f^{-1}(U)$ and $f^{-1}(U)$ is open as desired.

Now let K be compact and let $\{G_a\}$ be an open cover of $f(K)$. Since we just showed that the inverse image of any open set under f is open, we have that $\{f^{-1}(G_a)\}$ is an open cover of K, and since K is compact there is a finite subcover $\{f^{-1}(H_a)\}$ that covers K. Thus $\{f(f^{-1}(H_a))\} = \{H_a\}$ is a finite subcover of $\{G_a\}$ covering $f(K)$ and hence $f(K)$ is compact.

5. a) (6 points) Determine whether

$$\lim_{x\to 0}(\sin x)(\cos\frac{1}{x})$$

exists or not. If it exists, find its limit and prove that it is indeed the limit.

This limit exists and is 0. To prove this, let $\varepsilon > 0$ and let $\delta = \sin^{-1}(\varepsilon/2)$ in $(0, \pi/2)$ if $\varepsilon < 2$ and $\delta = \pi/3$ otherwise. Then if $x \neq 0$ and $|x - 0| = |x| < \delta$, we have $|(\sin x)(\cos\frac{1}{x}) - 0| = |(\sin x)(\cos\frac{1}{x})| < |\sin\delta| < \varepsilon$ as desired.

b) (4 points) Suppose $f(x)$ and $g(x)$ are real valued, continuous functions on $S \subset \mathbb{R}$, and let a be the limit of some sequence in S. Prove or provide a counterexample:

$$\lim_{\substack{x\to a\\ x\in S}} f(x)\cdot g(x) = (\lim_{\substack{x\to a\\ x\in S}} f(x))\cdot(\lim_{\substack{x\to a\\ x\in S}} g(x)).$$

This is true if the limits are all finite, but if, for example, $f(x) = x$, $g(x) = 1/|x|$, and $a = 0$, this is false.