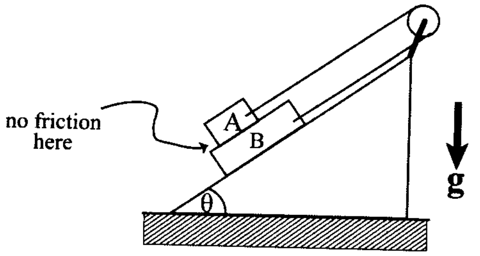
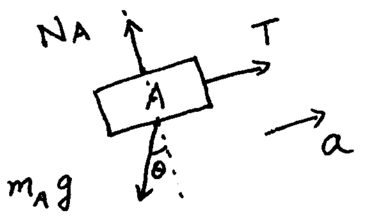
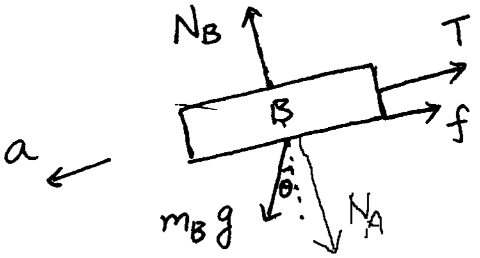
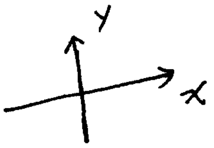
1. Block A (with mass m_A) slides on block B (with mass m_B), which slides on an inclined plane, as shown in the drawing below. The coefficient of kinetic friction between the inclined plane and block B is μ, but there's *no friction* between block A and block B. The string and pulley are massless. Also, we know that $m_B > m_A$.
   a) 5 Points. Draw a separate force diagram for block A and block B, showing all the forces.
   b) 10 Points. Assuming the blocks start from rest at $t = 0$, find the speed of block B down the inclined plane at arbitrary time t before block A slides off completely. Express your answer in terms of m_A, m_B, θ, μ, g, and t.
   c) 5 Points. Find the tension in the string connecting the blocks.



a)







b).

ΣF_{Ax} : $T - m_A g \sin\theta = m_A a$ ①

ΣF_{Ay} : $N_A - m_A g \cos\theta = 0 \Rightarrow N_A = m_A g \cos\theta$

ΣF_{Bx} : $m_B g \sin\theta - T - f = m_B a$ ②

ΣF_{By} : $N_B - N_A - m_B g \cos\theta = 0 \Rightarrow N_B = N_A + m_B g \cos\theta$
$= (m_A + m_B) g \cos\theta$

so $f = \mu N_B = \mu (m_A + m_B) g \cos\theta$

Add ① and ② :

$$(m_B - m_A) g \sin\theta - \mu (m_A + m_B) g \cos\theta = (m_A + m_B) a$$

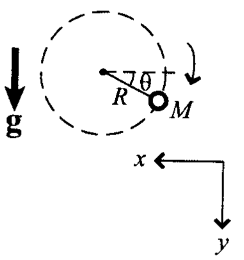
$$\Rightarrow a = \frac{m_B - m_A}{m_A + m_B} g \sin\theta - \mu g \cos\theta$$

const a $\Rightarrow$ $\boxed{V(t) = at = \left[\frac{m_B - m_A}{m_A + m_B} g \sin\theta - \mu g \cos\theta\right] t}$.

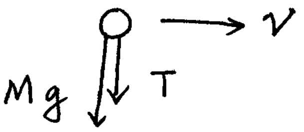
(c) From ① :

$$\boxed{T = m_A a + m_A g \sin\theta = m_A \left[\frac{m_B - m_A}{m_A + m_B} g \sin\theta - \mu g \cos\theta\right] + m_A g \sin\theta}$$

2. A ball of mass $M = 2$ kg at the end of a string of length $R = 2$ m revolves in a vertical circle as shown. The motion is circular but not uniform because of the force of gravity.
   a) 5 Points. Determine the minimum speed the ball must have at the highest point in the circle so the string doesn't slacken.
   b) 10 Points. Determine the direction and magnitude of the tangential acceleration, the radial acceleration, and the tension in the string at $\theta = 30°$ below the horizontal if the ball's speed is 6 m/s.
   c) 5 Points. If the string breaks at $\theta = 30°$ at $t = 0$, find the subsequent trajectory $y(t)$, $x(t)$ of the ball where $x = 0$ and $y = 0$ at $t = 0$.



a). Highest Point FBD :



$$F_c = Mg + T = M\frac{V^2}{R}.$$

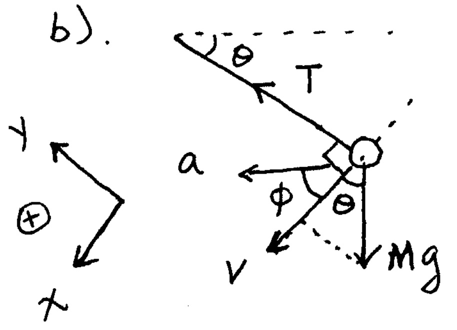
In order for tension T to be positive, i.e.

$$T = M\frac{V^2}{R} - Mg \geq 0 .$$

we must have $V \geq \sqrt{gR}$.

Hence the minimum speed is $\boxed{V_{min} = \sqrt{gR}}$.

b).



$$a_y = a_R = \frac{V^2}{R} = \boxed{18\ m/s^2}.$$ towards center

$$a_x = a_T = \frac{F_x}{M} = \frac{Mg\cos\theta}{M} = g\cos(30°)$$

$$\boxed{a_T = \frac{\sqrt{3}}{2}g} = \boxed{8.49\ m/s^2}$$ along V. or tangential.

~~$\phi = \tan^{-1}\left(\frac{a_y}{a_x}\right) = 64.7°$~~

~~The direction of the acceleration is at an angle of $\theta + \phi = 94.7°$ from the downward vertical to the left, or $\boxed{4.7°\ \text{above the left direction}}$.~~

$$\Sigma F_y : \ T - Mg\sin\theta = Ma_R \implies T = Ma_R + Mg\sin(\theta)$$

$$\boxed{T = 45.8\ N}$$

(c). $X_0 = Y_0 = 0$, $V_{ox} = -V\sin\theta = +3\ m/s$ $V_{oy} = +V\cos\theta = +5.2\ m/s$

$a_x = 0$, $a_y = +g$.

$$x(t) = x_0 + V_{ox}t + \tfrac{1}{2}a_x t^2 = \boxed{+3t}$$

$$y(t) = y_0 + V_{oy}t + \tfrac{1}{2}a_y t^2 = \boxed{+5.2t + \tfrac{1}{2}gt^2}.$$

Problem 3

(a) Derivation of I for cylinder

$I = \int r^2 dm$ consider cylindrical shells

$dm = 2\pi L \rho r dr$ where ρ = mass densi

$\rho = M/\pi R^2 L$

$dm = \frac{2Mrdr}{R^2 \cancel{L}}$

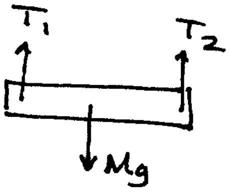
$$I = \int_0^R r^2 \frac{2Mrdr}{R^2 \cancel{L}}$$

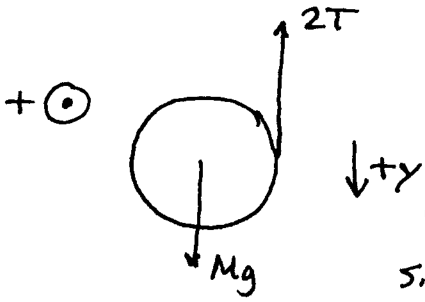
$$= \frac{2M}{R^2 \cancel{L}} \int_0^R r^3 dr$$

$$= \frac{2M}{R^2} \frac{r^4}{4} \Big|_0^R$$

$$\boxed{I = \frac{1}{2} MR^2}$$

We know the tension in each cord is the same if we draw the FBD for the side view. Torque contributions about the center must cancel so $T_1 = T_2 = T$





Define down and out of the page to be positive

① $\Sigma F_y = Mg - 2T = Ma \Rightarrow a = g - \frac{2T}{M}$

② $\Sigma \tau = 2T \cdot R = I\alpha$

Since we know it unwinds w/o slipping, we can relate the linear and angular accelerations $\Rightarrow a = R\alpha$

② $2TR = I \frac{a}{R}$ plug in ① $\Rightarrow$

$$2TR^2 = I\left(g - \frac{2T}{M}\right) \Rightarrow T = \frac{IgM}{2R^2M + 2I}$$ plug in $I = \frac{1}{2}MR^2$

$$T = \frac{\frac{1}{2}M^2R^2g}{2R^2M - MR^2}$$

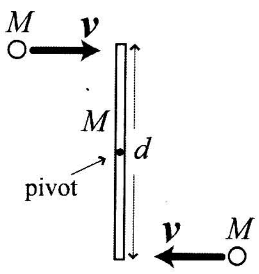
$$\boxed{T = \frac{1}{6}Mg}$$

(b) Plug in this tension into eq ①

$$a = g - \frac{2T}{M} = g - \frac{2}{M}\left(\frac{1}{6}Mg\right)$$

$$\boxed{a = \frac{2}{3}g}$$

4. Two particles, each of mass M and speed v, move as shown. They simultaneously strike the ends of a uniform rod of mass M and length d which is pivoted at its center. The particles stick to the ends of the rod.
   a) 5 Points. Find the magnitude and direction of the angular momentum with respect to the center of the rod before the collision.
   b) 5 Points. What is the angular momentum after the collision? Justify your answer.
   c) 10 Points. Find the angular speed of rotation of the particles and the rod after the collision.
   d) 5 Points. How much kinetic energy is lost in the collision of the two particles with the rod? Where does the energy go?



(a) $L = I\omega = mr^2 \frac{v}{r} = mvr$. our $r = \frac{d}{2}$ $\Rightarrow L = \frac{mvd}{2}$ ⊗ via RHR

~or~

$\vec{L} = \vec{r} \times \vec{p} = \frac{d}{2} mv$ ⊗ into page

we have 2 particles $\Rightarrow \vec{L}_{Total} = 2\vec{L} = \boxed{mvd \otimes}$

(b) No external Torques on particle / rod system

$\Rightarrow \vec{L}$ conserved $\Rightarrow L_f = \boxed{L_i = mvd \otimes}$

(c) $I_{rod} = \frac{1}{12} M d^2$ $\quad 2 I_{particles} = 2mr^2 = 2m\left(\frac{d}{2}\right)^2 = \frac{md^2}{2}$

$I^{(cm)}_{total\,after} = I_{rod} + 2 I_{part} = \left(\frac{1}{12} + \frac{6}{12}\right) Md^2 = \boxed{\frac{7}{12} Md^2}$

~~d~~)

$L = I\omega$

$\omega = \frac{L}{I} = \frac{mvd}{\frac{7}{12} md^2} = \boxed{\frac{12}{7}\frac{v}{d} = \omega}$

(d) $KE_i = \frac{1}{2}mv^2 + \frac{1}{2}mv^2 = mv^2$

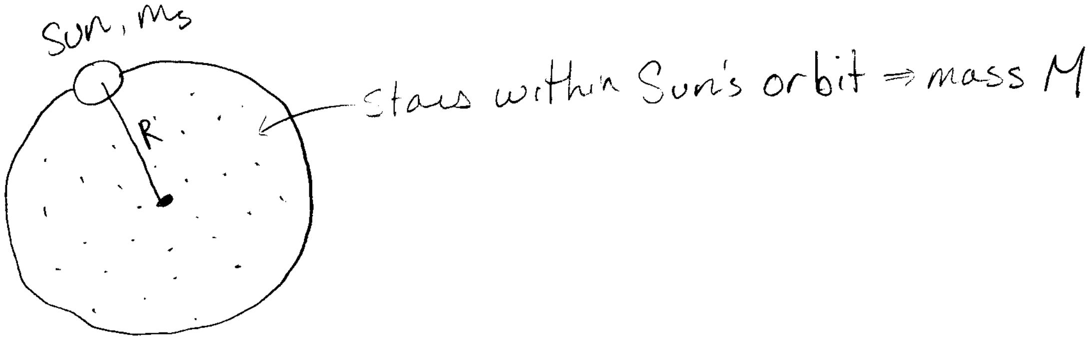
$KE_f = \frac{1}{2} I_{cm} \omega^2 = \frac{1}{2}\left(\frac{7}{12} Md^2\right)\left(\frac{12}{7}\frac{v}{d}\right)^2 = \frac{6}{7} Mv^2$

$KE_{lost} = KE_i - KE_f = mv^2 - \frac{6}{7} mv^2 = \boxed{\frac{1}{7} mv^2}$

Goes into sticking
heat
sound

5. Suppose the Sun is traveling with velocity 220 km/s in a circular orbit of radius 2.5×10^{17} km about the center of our galaxy. ($G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$)
   a) 10 Points. If the mass distribution of the galaxy is spherically symmetric about the center, find the total galactic mass contained inside the orbit of the Sun.
   b) 10 Points. Assuming there is no additional mass outside the orbit of the Sun, derive how fast an object at that distance would have to travel to escape from the galaxy.

a)



$$F_G = \frac{GMm_s}{R^2} = \frac{m_s v^2}{R}$$

$$\Rightarrow M = \frac{v^2 R}{G} = \frac{(220\cdot 10^3\, m/s)^2 (2.5\cdot 10^{20} m)}{6.67\cdot 10^{-11}\, Nm^2/kg^2}$$

$$= \boxed{1.81\cdot 10^{41}\, kg}$$

b)

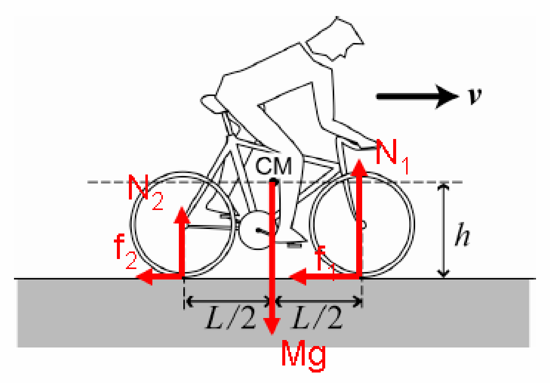
$$E_i = \frac{1}{2} m v_{esc}^2 - \frac{GMm}{R} = E_f = 0$$

$$\text{so } v_{esc} = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2\cdot 6.67\cdot 10^{-11}\frac{Nm^2}{kg^2}\cdot 1.81\cdot 10^{41} kg}{2.5\cdot 10^{20} m}}$$

$$= 3.1\cdot 10^5\, m/s = \boxed{310\, km/s}$$

6. The key issue of this problem is that when there's linear acceleration, the only choice of pivot point is the center of mass (CM). In a statistics problem, we have the freedom to choose a pivot point, but here because of the deceleration, we lose that freedom. In other words, we have to decompose the motion of the system to two parts: the translational motion of CM and the rotational motion around CM. This is important because we need to know the pivot point before we calculate all the torques.

a) After the brakes are applied, the force diagram is as following:



There's no acceleration in the vertical directions, so I have

$$N_1 + N_2 = Mg \qquad (1)$$

There's no rotational acceleration around CM, which gives rise to

$$N_1 * \frac{L}{2} - \mu_k N_1 * h - N_2 * \frac{L}{2} - \mu_k N_2 * h = 0 \qquad (2)$$

Here I've already used the relation $f_1 = \mu_k N_1$ and $f_2 = \mu_k N_2$.

From equation (1), $N_2 = Mg - N_1$. Then plug it into equation (2),

$$N_1 * \left(\frac{L}{2} - \mu_k h\right) = N_2 * \left(\frac{L}{2} + \mu_k h\right) = (Mg - N_1) * \left(\frac{L}{2} + \mu_k h\right)$$

Therefore, $N_1 = \frac{L + 2\mu_k h}{2L} Mg$ & $N_2 = \frac{L - 2\mu_k h}{2L} Mg$.

At the same time, $f_1 = \frac{L + 2\mu_k h}{2L} \mu_k Mg$ & $f_2 = \frac{L - 2\mu_k h}{2L} Mg$

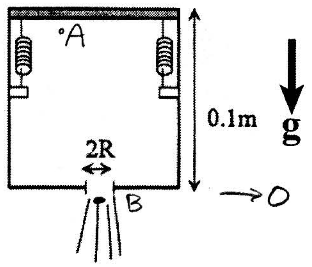
b) The critical condition for the overturning to happen is the normal force N_2 becomes zero.

$$N_2 = \frac{L - 2\mu_k h}{2L} Mg = 0 \quad ,$$

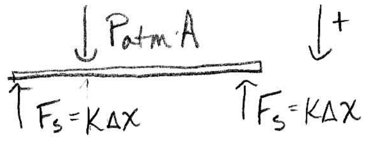
Then $\frac{L}{2h} = \mu_k$.

The condition to prevent that from happening is $\frac{L}{2h} > \mu_k$.

7. A water "rocket" consists of a cubic chamber of side length $L = 0.1$ m that is filled with water (neglect the mass of the chamber walls relative to the water). The lid (also of negligible mass) is held on with two (stiff) springs of spring constant $k = 10^7$ N/m, which are stretched a distance 0.05 m from their equilibrium extension. A small circular hole of radius R is on the bottom of the box through which the water can escape. (1 atmosphere = 10^5 Pa. Density of water = 10^3 kg/m^3)
   a) 5 Points. What is the pressure just under the lid of the box?
   b) 10 Points. What is the velocity of the liquid leaving the box?
   c) 10 Points. How large must the radius R of the hole be for the water rocket to just get off the ground? (Hint: The flow of momentum out of the box is $dP/dt = d(mv)/dt = v\, dm/dt$, where v is the velocity of the escaping water and dm/dt is the mass of water per unit time leaving the box.)



a)



$$\Sigma F_y = P_{atm} A + 2F_s = F_{NET}$$

$$P = F_{NET}/A = P_{atm} + \frac{2k\Delta x}{A}$$

$$= \frac{10^5 Pa + 2\cdot 10^7 \frac{N}{m}\cdot 0.05m)}{(0.1)^2} = \boxed{10^8 pa}$$

b) Setting the origin at the bottom, Bernoulli's equation gives us

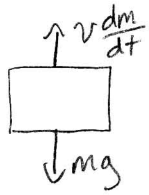
$$P_A + \rho g h + \cancelto{0}{\tfrac{1}{2}\rho v_A^2} = P_{atm} + \cancel{\rho g 0} + \tfrac{1}{2}\rho v_B^2$$

Top of tank — Just below hole

$$\Rightarrow v_B^2 = \frac{(P_A - P_{atm}) + \rho g h}{\rho/2} = \frac{(10^8 Pa - 10^5 Pa) + (1000 kg/m^3 \cdot 10 m/s^2 \cdot 0.1m)}{\frac{1000 kg/m^3}{2}}$$

$$= 2\cdot 10^5 \frac{m^2}{s^2} \Rightarrow \text{So } v_B = \sqrt{2\cdot 10^5} = \boxed{447 m/s}$$

c) $\Sigma F_y > mg$ to get off ground



$$v\frac{dm}{dt} = mg \qquad \frac{dm}{dt} = \pi R^2 v_B, \quad m = V\rho_{water}$$

$$\Rightarrow v_B \pi R^2 v_B = V\rho_{water} g$$

$$R = \sqrt{\frac{V\rho_{water} g}{\pi v_B^2}} = \sqrt{\frac{(0.1m)^3 \cdot 1000 Kg/m^3 \cdot 10 m/s^2}{\pi \cdot 447^2 \frac{m^2}{s^2}}} = \boxed{0.004m}$$

8. Solution.

Before collision, the motion is described by $M\ddot{X} = -kX$ and the solution is $X = x_0 \sin(\omega t - \phi_0)$, $V = x_0\omega \cos(\omega t - \phi_0)$, $\omega = \sqrt{\frac{k}{M}}$.
The mass M has its maximum velocity at its equilibrium point

$$\underline{V_{max} = X_0 \sqrt{k/M}}$$

During the inelastic collision, the momentum is conserved

$$mv + MV_{max} = (m+M)V' \rightarrow \underline{V' = \frac{1}{m+M}(mv + x_0\sqrt{kM})}.$$

This is the new maximum velocity of new system with mass $(m+M)$ and spring constant $k \rightarrow (m+M)\ddot{X}_f = -kX_f$.
New motion can be described by $X_f = X_0' \sin(\omega' t - \phi_0')$,
here $\underline{\omega' = \sqrt{\frac{k}{m+M}} = 2\pi f'}$ $\therefore \underline{\underline{f' = \frac{1}{2\pi}\sqrt{\frac{k}{m+M}}}}$.

The new maximum velocity is given by

$$\dot{X}_f\Big|_{max} = X_0'\omega' \cos(\omega' t - \phi_0')\Big|_{max} = X_0'\omega' = X_0'\sqrt{\frac{k}{m+M}}.$$

This maximum velocity should be equal to V'

$$\therefore \quad V' = \dot{X}_f\Big|_{max} \Rightarrow \frac{mv + x_0\sqrt{kM}}{m+M} = X_0' \cdot \sqrt{\frac{k}{m+M}}.$$

$$\Rightarrow \quad \underline{\underline{X_0' = \frac{mv + x_0\sqrt{kM}}{\sqrt{k(m+M)}}}} \quad \text{(new Amplitude)}$$

$$\underline{\underline{f' = \frac{1}{2\pi}\sqrt{\frac{k}{m+M}}}} \quad \text{(new frequency)}$$

9. One end ($x = 0$) of a rope of mass per unit length, μ, is moved up and down in simple harmonic motion $y = y_0 \sin \omega t$.

a) 6 Points. If the rope is under tension F_T, find the wavelength, frequency, and phase speed in terms of the given quantities.

b) 4 Points. Find the function of x and t that describes the wave that results.

c) 5 Points. Find the total mechanical energy (potential plus kinetic) per unit length, and the power carried by the wave past a fixed point.

d) 10 Points. Suppose the rope can be fixed to $y = 0$ at $x = L$. Find an expression for the values of L for which standing waves are set up.

a) $y(x=0,t) = y_0 \sin(\omega t)$ μ, y_0, ω and F_T are given.

wave moves to the right: $y(x,t) = y_0 \sin(-kx + \omega t)$

$\boxed{\text{phase speed: } v = \sqrt{\frac{F_T}{\mu}}}$ velocity: $\vec{v} = \sqrt{\frac{F_T}{\mu}}\,\hat{x}$

$v = \lambda f$ $\boxed{f = \frac{\omega}{2\pi} \text{ is the frequency}}$

$\boxed{\lambda = \frac{v}{f} = \frac{2\pi}{\omega}\sqrt{\frac{F_T}{\mu}} \text{ is the wavelength}}$

b) $y(x,t) = y_0 \sin(\omega t - kx)$ with $k = \frac{2\pi}{\lambda} = \frac{\omega}{v} = \omega\sqrt{\frac{\mu}{F_T}}$

$\boxed{y(x,t) = y_0 \sin\left(\omega t - \omega\sqrt{\frac{\mu}{F_T}}\, x\right)}$ describes the wave

c) Energy per unit length:

Kinetic: $K/\ell = \frac{\mu}{2}\left(\frac{\partial y}{\partial t}\right)^2$

$K/\ell = \frac{\mu\omega^2}{2} y_0^2 \cos^2(\omega t - kx)$

Potential: $U/\ell = \frac{\mu\omega^2 y^2}{2}$

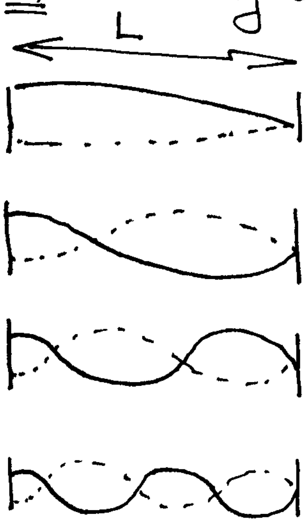
$U/\ell = \frac{\mu\omega^2}{2} y_0^2 \sin^2(\omega t - kx)$

Total: $\boxed{\frac{E}{\ell} = \frac{K}{\ell} + \frac{U}{\ell} = \frac{\mu\omega^2 y_0^2}{2}}$

Power: $P = \frac{E}{t} = \frac{E}{\ell}\frac{\ell}{t} = \frac{E}{\ell} v$

$\boxed{P = \frac{1}{2}\sqrt{F_T \mu}\,\omega^2 y_0^2}$

d) Standing Waves with one end moving!



$L = \frac{\lambda}{4}$

$L = \frac{\lambda}{4} + \frac{\lambda}{2}$

$L = \frac{\lambda}{4} + \lambda$

$L = \frac{\lambda}{4} + \frac{3\lambda}{2}$

In general $L = \frac{\lambda}{4} + n\frac{\lambda}{2}$

$L = \left(\frac{1}{4} + \frac{n}{2}\right)\frac{2\pi}{\omega}\sqrt{\frac{F_T}{\mu}}$ with integer n.