MATH 104 MIDTERM 1 solutions

1. (2 points each) Mark each of the following as True or False. No justification is required.

a) Every Cauchy sequence of real numbers is bounded. **True**

b) There is a sequence whose set of subsequential limits is $[0,1]$. **True**

c) Let $\{s_n\}$ be a sequence of real numbers such that the set $\{n \in \mathbb{N} \mid |s_n - 3| < \varepsilon\}$ is infinite for any $\varepsilon > 0$. Then there is a monotone subsequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $\liminf s_{n_k} = 3$. **True**

d) If A is a Dedekind cut then $\{2a \mid a \in A\}$ is a Dedekind cut. **True**

e) A sequence of real numbers which has a subsequence whose limit is 1.04 has a monotone subsequence whose limit is 1.04. **True**

2. a) (6 points) Show that the sequence $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$ converges.

We show that this sequence is increasing and bounded above by 2, hence convergent. Specifically, we show by induction that $2 \geq s_n \geq s_{n-1}$ for all n. The base case $n = 2$ is true: $2 \geq 2\sqrt{2} \geq \sqrt{2}$. Suppose that for some $k \geq 2$ we have $2 \geq s_k \geq s_{k-1}$. Then $s_{k+1} = \sqrt{2s_k} \geq \sqrt{s_k \cdot s_k} = s_k$ since $2 \geq s_k$. Also, for the same reason, $s_{k+1} = \sqrt{2s_k} \leq \sqrt{2 \cdot 2} = 2$. So we have that $2 \geq s_{k+1} \geq s_k$ and hence by induction what we have claimed is true.

b) (4 points) Find the limit of the sequence in part (a).

Since the limit exists by part (a), we may write $a = \lim s_n = \lim s_{n+1} = \lim \sqrt{2s_n}$ and hence $a^2 = \lim 2s_n$ (since the product of limits of two sequences is the limit of the product of the sequences) and so we have $a^2 = 2a$ and so $a = 0$ or 2. But a cannot be 0 since all of the terms in the sequence are $\geq \sqrt{2}$. Hence the limit is 2.

3. (10 points) Let $\{s_n\}$ be a sequence of real numbers, bounded below by 104 and above by 2014. Show that

$$\liminf \frac{1}{(s_n)^2} = \frac{1}{(\limsup s_n)^2}$$

Let S be the set of subsequential limits of s_n. Since $104 \leq s_n \leq 2014$ for all n, we have that S is bounded below by 104 and above by 2014. Let $t_n = 1/(s_n)^2$.

We show that a subsequence t_{n_k} of t_n converges iff s_{n_k} converges. The sequence t_{n_k} is always of the form $1/(s_{n_k})^2$, and if it converges then $1/t_{n_k} = (s_{n_k})^2$ converges as well (note that $t_{n_k} > 1/2014$ for all k and so the sequence is nonzero and has a nonzero limit). Since $(s_{n_k})^2$ converges then so does s_{n_k}: if s_{n_k} does not converge then there is some ε such that for all N we have $|s_{n_K} - s_{n_L}| > \varepsilon$ for some $K, L > N$, and hence $|s_{n_K}^2 - s_{n_L}^2| > \varepsilon/|s_{n_K} + s_{n_L}| > \varepsilon/4028$ for all $K, L > N$ and hence $(s_{n_k})^2$ does not converge either.

For the other direction, if s_{n_k} converges, it converges to a nonzero limit and all of its terms are nonzero. Hence $t_{n_k} = 1/(s_{n_k})^2$ converges as well.

Finally, note that neither s_n nor t_n has a subsequence diverging to ∞ or $-\infty$ since the sequences are bounded.

This implies that the set of subsequential limits of t_n is simply $\{1/a^2 \mid a \in S\}$. The infinimum of this set is the reciprocal of the square of the supremum of S, since S consists of positive real numbers only. Hence, indeed, $\liminf \frac{1}{(s_n)^2} = \frac{1}{(\limsup s_n)^2}$.

4. a) (5 points) Let $n \in \mathbb{Z}$, and let $k \in \mathbb{N}$ such that n is not a k-th power of any integer. Show that $\sqrt[k]{n}$ is irrational.

$a = \sqrt[k]{n}$ satisfies the polynomial equation $x^k - n = 0$. By the rational roots test, the only rational numbers satisfying this equation are such that the numerator divides n, and the denominator divides 1. Hence any rational root of this polynomial is also an integer. But we know that n is not the k-th power of any integer, so $\sqrt[k]{n}$ cannot be rational.

b) (5 points) Show that $a = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}$ is irrational.

Suppose a is rational. Then, since $-2 \in \mathbb{Q}$ and $\mathbb{Q}$ is closed under multiplication and addition, $\sqrt{2} = (((a^2-2)^2-2)^2-2)^2-2 \in \mathbb{Q}$ as well, but we have shown that $\sqrt{2} \notin \mathbb{Q}$. Hence $a \notin \mathbb{Q}$ either.

5. (10 points) Show that $\left\{\frac{2n+1}{n}\right\}$ is a Cauchy sequence, using the definition of a Cauchy sequence only.

Let $\varepsilon > 0$.

$$\left|\frac{2n+1}{n} - \frac{2m+1}{m}\right| = \left|2 - 2 + \frac{1}{n} - \frac{1}{m}\right| = \left|\frac{m-n}{mn}\right| < \varepsilon \text{ for all } m, n > \frac{1}{\varepsilon}.$$

Hence the sequence is Cauchy.