Physics H7C; Midterm 2

Tuesday, 4/8; 9 AM -10:30 AM

Write your responses below, or on extra paper if needed. Show your work, and take care to explain what you are doing; partial credit will be given for incomplete answers that demonstrate some conceptual understanding. Cross out or erase parts of the problem you wish the grader to ignore.

Problem 1: Appetizers (5 pts each)

1a) As you race to class on your bike, your tires spin at 180 RPMs (rotations per minute). If an electron is stuck at some point on a wheel, at what wavelength is the emitted electromagnetic radiation?

1b) Consider two monochromatic plane waves, described using complex representation

$$\vec{E}_1 = E_0 \hat{y} e^{i(kx-\omega t)} \quad \text{and} \quad \vec{E}_2 = E_0(1+i)\hat{y}e^{i(kx-\omega t)} \tag{1}$$

where E_0 is a real number. What is the ratio of the intensities of wave 2 and wave 1? What is the relative phase difference between the two waves?

1c) An object is placed extremely far away from a single lens. For each of two lenses at right, say whether the resulting image would be real or virtual, and whether it would be inverted or upright. Explain your answers in terms of the curvatures of the lenses.

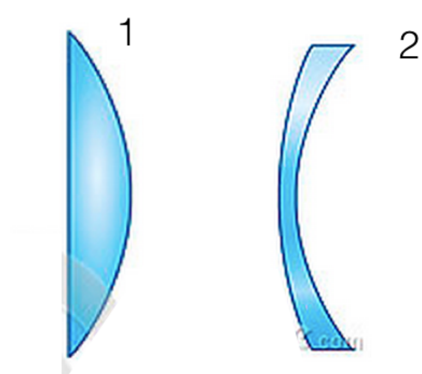


Figure 1:

Problem 2: Take a Look at Yourself (20 pts)

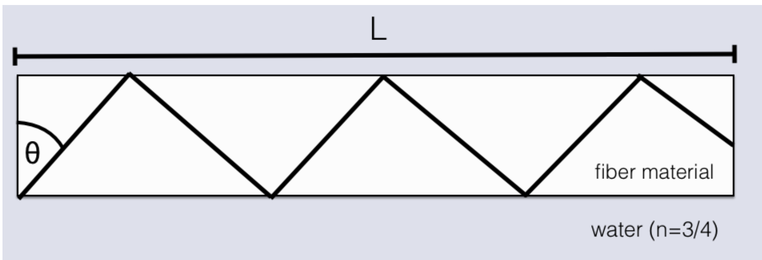
YOU VISIT THE LAWRENCE HALL OF SCIENCE, and stare directly into their giant concave mirror. As you move nearer and farther from the mirror, you find a location a distance D from the mirror vertex at which the height of your image is exactly the same as your height.

2a) As you move around some more, you find that there are two locations where your image height is twice your height. What are these two distances, in terms of D?

2b) How is the distance D related to the focal length of the mirror?

Problem 3: Fiber optic (20 pts)

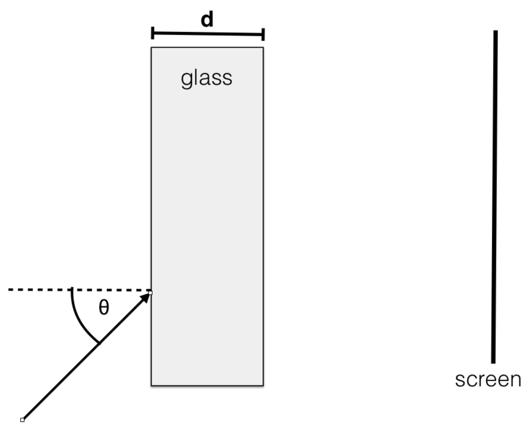
A FIBER OPTIC CABLE of length L is submerged under the ocean. As seen in the figure, a light ray is emitted from the left bottom corner at an angle θ. The angle θ is chosen such that the time it takes the ray to propagate all the way down the fiber is a maximal value, $t_{\max}$.



3a) Given measurements of L and $t_{\max}$, find an expression for the index of refraction of the optical fiber material.

Problem 4: Rainbow through the Window (20 pts)

A NARROW, COLLIMATED BEAM OF white light is incident on a glass window at an angle θ. After passing through the glass, the beam lands on a screen and forms a rainbow image.

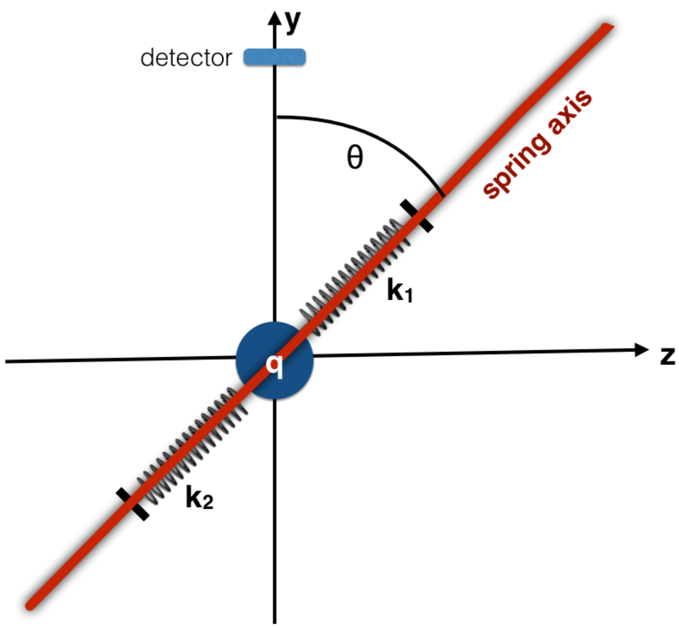


4a) What is the orientation of the rainbow – that is, what color of the rainbow is nearer the top of the screen? Explain your answer.

4b) Find an expression for the height of the rainbow image on the screen, in terms of θ, the window thickness d, and the indices of refraction. n_r and n_v, of red and violet light respectively.

Problem 5: Springy Charge (25 pts)

A CHARGED PARTICLE of mass m and charge q is connected on either side by springs, with two different spring constants k_1 and k_2. The particle is constrained to move only along the spring axis, and not perpendicular to it.



A monochromatic plane wave polarized in the y-direction and moving in the x-direction (into the plane of the paper) is incident on the particle. The plane wave has a wavelength λ_0 which is much larger than the dimensions of the system.

5a) If the spring axis is oriented along the y-axis ($\theta = 0°$) what is the total power radiated (in all directions) by the particle as a function of time?

5b) If the spring axis is oriented along the z-axis ($\theta = 90°$) what is the total power radiated by the particle (in all directions) as a function of time?

5c) A light detector is placed on the y-axis, far away from the particle/spring system. How does the intensity measured by the detector change as you vary the angle θ ? You can write this as $I(\theta) \propto f(\theta)$, where $f(\theta)$ is the functional dependence on θ.

extra work space

Midterm 2 Solutions

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1. (a) The oscillations of the emitted radiation should correspond to full rotations of the electron around the wheel. Hence the frequency of the electromagnetic wave is the same as the frequency of rotation of the wheel. We have

$$\begin{aligned}\lambda &= \frac{c}{f} \\ &= \frac{3 \times 10^8 \text{ m/s} \times 60 \text{ s/min}}{180 \,\text{min}^{-1}} \\ &\approx 10^8 \text{ m} \quad . \end{aligned} \tag{1}$$

(b) In the complex plane, where the reals lie on the x-axis, and the imaginaries lie on the y-axis, the amplitude of wave 1 looks like a vector $(1,0)$ and the amplitude of wave 2 one with $(1,1)$. We see that the magnitude of E_1 is 1, and that of $E_2 = \sqrt{2}$. The intensity of a wave goes like the electric field squared, so the ratio $I_2/I_1 = 2$. The polar angle between them is 45° or $\pi/4$.

More formally, the intensity of a wave is proportional to the amplitude of the wave squared

$$I \propto E^* E \tag{2}$$

where E^* is the complex conjugate of the field E. Hence

$$\frac{I_2}{I_1} = \frac{E_0(1+i)E_0(1-i)}{E_0^2} = \frac{E_0^2(1-i^2)}{E_0^2} = 2 \tag{3}$$

We find the argument (polar angle in the complex plane) of each wave to determine the phase shift. The exponential portion of the wave is the same in each case, so we look only at the phase of the amplitudes. The first wave has only a real part, and hence has complex argument 0. The second wave has argument $\arctan 1 = \pi/4$, hence the second wave is shifted $\pi/4$ relative to the first.

(c) We use the lens makers equation to determine whether each lens is converging or diverging

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{4}$$

Taking the object to be to infinitely far away to the left, lens 1 has $R_1 = \infty$ and $R_2 < 0$. It therefore has a positive focal length and is converging. A converging lens produces an image that's inverted and real. Lens 2 has $R_1 > 0$ and $R_2 > 0$. We can see that the absolute value of R_2 is less than R_1, therefore the lens makers equation gives us a negative focal length. This is therefore a diverging lens and produces an image that's right side up and imaginary.

2. (a) The transverse magnification is given by $M = s_i/s_o$, and we have the mirror equation $1/s_i + 1/s_o = 1/f$, so taking $s_o = D$

$$\begin{aligned} M &= \frac{f}{f-D} \\ &= \pm 1 \\ \Longrightarrow D &= 2f \text{ or } 0 \quad . \end{aligned} \tag{5}$$

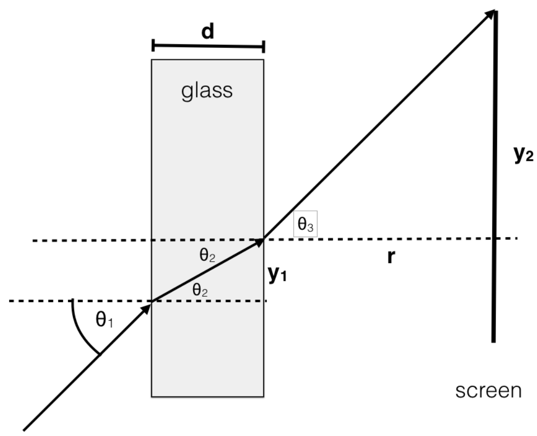
We consider both ± 1 is because we allow the image to be either inverted (and real) or upright (and virtual). Here we ignore the extraneous solution $D = 0$. Now considering the case where $M = 2$

$$\begin{aligned} M &= \frac{f}{f-D'} \\ &= \pm 2 \\ \Longrightarrow D' &= \frac{1}{2}f \text{ or } \frac{3}{2}f \\ &= \frac{1}{4}D \text{ or} \frac{3}{4}D \quad . \end{aligned} \tag{6}$$

(b) from above, we see that $f = D/2$.

3. (a) If the ray takes the maximum time to traverse the cable, the angle θ must be the smallest possible while still permitting total internal reflection. This means θ is the critical angle, defined by $\sin\theta_c = n_w/n_c$, where n_c and n_w are the indices of refraction of the cable and of water. The geometry of the problem (and the law of reflection) implies that the path length traveled by the ray is $L/\sin\theta_c$. The speed of the light ray in the cable is $v = c/n_c$. Thus

$$t_m = \frac{L}{\sin\theta_c v} = \frac{L}{\sin\theta_c (c/n_c)} = \frac{L}{(n_w/n_c)(c/n_c)} = \frac{Ln_c^2}{cn_w} \tag{7}$$



and solving for n_c

$$n_c = (t_m c n_w / L)^{1/2} \tag{8}$$

4. (a) Glass has a higher index of refraction at bluer wavelengths. The rainbow thus appears with red at the top, and blue at the bottom, since shorter wavelengths tend to be refracted more strongly towards the horizontal when entering the glass. See figure.

   (b) Looking at the figure, we see application of Snell's law gives $\sin\theta_1 = n\sin\theta_2$ and a second application shows $\theta_3 = \theta_1$. The vertical position is $y = y_1 + y_2 = d\tan\theta_2 + r\tan\theta_1$. The difference in y between red and violet light is then

$$\Delta y = y_r - y_v = d\tan\theta_{2,r} - d\tan\theta_{2,v} \tag{9}$$

   The distance to the screen r, cancels out. One could write the solution directly

$$\Delta y \quad = \quad d\left(\tan\left(\arcsin\left(\frac{1}{n_r}\sin\theta_1\right)\right) - \tan\left(\arcsin\left(\frac{1}{n_v}\sin\theta_1\right)\right)\right) \tag{10}$$

   But to make things more beautiful we can write

$$d\tan\theta_2 = \frac{d\sin\theta_2}{\cos\theta_2} = \frac{d\sin\theta_2}{(1-\sin^2\theta_2)^{1/2}} \tag{11}$$

$$= \frac{d\sin\theta_1/n}{(1-\sin^2\theta_1/n^2)^{1/2}} = \frac{d\sin\theta_1}{(n^2-\sin^2\theta_1)^{1/2}} \tag{12}$$

   And so the image size is

$$\Delta y = d\sin\theta_1[(n_r^2 - \sin^2\theta_1)^{-1/2} - (n_v^2 - \sin^2\theta_1)^{-1/2}] \tag{13}$$

5. (a) As in our analysis of the blue sky (from the problem set), we can write an equation of motion of a charged particle with a harmonic force

$$m\ddot{s} = -k_1 s - k_2 s + qE_0 \cos\theta e^{-i\omega t} \tag{14}$$

where s is the displacement along the spring axis. The $E_0 \cos\theta$ term is the component along the spring axis of the incident electric field.

$$\ddot{s} = -\omega_0^2 s + \frac{qE_0 \cos\theta}{m} e^{-i\omega t} \tag{15}$$

where $\omega_0 = \sqrt{(k_1 + k_2)/m}$ is the resonant oscillation frequency. We anticipate a solution $s(t) = s_0 e^{-i\omega t}$ and plugging this in find

$$-\omega^2 s_0 = -\omega_0^2 s_0 + \frac{qE_0 \cos\theta}{m} \tag{16}$$

and so we find the motion

$$s(t) = \frac{qE_0 \cos\theta}{m(\omega_0^2 - \omega^2)} e^{-i\omega t} \tag{17}$$

the acceleration is then

$$a = \ddot{s} = -\omega^2 \frac{qE_0 \cos\theta}{m(\omega_0^2 - \omega^2)} \cos(\omega t) \tag{18}$$

where we took the real part of the complex exponential. The Larmor formula gives

$$P = \frac{q^4 E_0^2 \cos^2\theta \omega^4}{6\pi\epsilon_0 c^3 m^2 (\omega_0^2 - \omega^2)^2} \cos^2(\omega t) \tag{19}$$

For part a) the angle is $\theta = 0$, so this simply becomes

$$P = \frac{q^4 E_0^2 \omega^4}{6\pi\epsilon_0 c^3 m^2 (\omega_0^2 - \omega^2)^2} \cos^2(\omega t) \tag{20}$$

(b) In this case, the electric field is perpendicular to the spring axis and so doesn't move the electron, and hence the radiated power is zero.

(c) As seen above, the electromagnetic force (and hence the acceleration) the electron feels is proportional to $\cos\theta$ where θ is the angle between the spring axis and the electric field vector. Hence the total power is thus proportional to $\cos^2\theta$. In addition, an oscillating particle radiates with a $\sin^2\theta$ distribution pattern. Therefore the intensity measured along the y-axis has the functional dependence

$$I(\theta) \propto \cos^2\theta \sin^2\theta \tag{21}$$