# *Physics H7C; Midterm 1 Solutions*

*Tuesday, 2/25; 9 AM -10:30 AM*

**Problem 1: Pion Goes the Distance (20 pts)**

A $\pi^+$ IS AN UNSTABLE PARTICLE *that can decay into a neutrino and a muon. It's mean lifetime as measured when at rest is* $25 \times 10^{-9}$ *seconds (i.e., 25 nanoseconds). Its rest mass energy is* $140\ \mathrm{MeV/c^2}$.

**1a)** *An experiment carried out in the end zone of Cal Memorial Stadium creates a pion moving at very near the speed of light. It is seen to travel 75 meters (almost a touchdown) before it decays. What is the Lorentz factor of the pion?*

Because the pion is moving, the time (call it $\Delta t$) it takes the pion to decay in the lab frame is time dilated relative to the decay time in the rest frame, $\tau$ (the proper time)

$$\Delta t = \gamma\tau \tag{1}$$

The time elapsed in the lab is $\Delta t = D/v$ where $D = 75$ m and $v$ is the pion velocity. Approximating $v \approx c$ and solving for $\gamma$ then gives

$$\gamma = D/v\tau \approx D/c\tau = \frac{75}{(3\times10^8)\times(25\times10^{-9})} = 10 \tag{2}$$

Since $\gamma$ is quite a bit greater than 1, the approximation that $v \approx c$ is a reasonable one.

If you don't want to make the approximation $v \approx c$, you can solve the problem exactly. Squaring the above and using the relation $v^2/c^2 = 1 - \gamma^{-2}$

$$\gamma^2 = \frac{D^2}{v^2\tau^2} = \frac{D^2}{c^2\tau^2(1-\gamma^{-2})} \tag{3}$$

Solving for $\gamma$ with a little algebra gives

$$\gamma^2(1-\gamma^{-2}) = (D/c\tau)^2 \tag{4}$$

$$\gamma^2 - 1 = (D/c\tau)^2 \tag{5}$$

$$\gamma^2 = (D/c\tau)^2 + 1 \tag{6}$$

$$\gamma = [(D/c\tau)^2 + 1]^{1/2} \tag{7}$$

Plugging in the fact above that $D/c\tau = 10$, we get

$$\gamma = (101)^{1/2} \approx 10 \tag{8}$$

If you really want to estimate how far off the approximation was, try an expansion

$$\gamma = (101)^{1/2} = 10(1+1/100)^{1/2} = 10(1+1/200+\ldots) \tag{9}$$

So the approximation $v \approx c$ only leads to an error of order about 0.5%.

**1b)** *What is the energy (in MeV) of this pion? What is it's momentum (in MeV/c)?*

The energy is given by $E = \gamma m_\pi c^2 = 1400\ \mathrm{MeV/c^2}$.

Since the pion velocity is $v \approx c$ (or equivalently, $E_\pi \gg m_\pi c^2$) one can approximate the pion as massless and just say $p = E/c = 1400\ \mathrm{MeV/c}$.

However, more accurately the momentum for a massive particle is given by

$$(pc)^2 = E^2 - (m_\pi c^2)^2 = (\gamma m_\pi c^2)^2 - (m_\pi c^2)^2 \tag{10}$$

and so

$$pc = \sqrt{(m_\pi c^2)^2(\gamma^2 - 1)} = m_\pi c^2 \sqrt{\gamma^2 - 1} \tag{11}$$

$$p = (m_\pi c)\sqrt{99} \approx 10(m_\pi c^2) = 1400\ \mathrm{MeV/c} \tag{12}$$

**Problem 2: The Battle Continues (25 pts)**

THE OUTCOME OF THE NOTORIOUS *Einstein/Bohr footrace – where the two started at the same point and ran in opposite directions – was hotly contested. So the two decided to turn around and race back. They each started on opposite sides Marie Curie, an equal distance, $D$, away from her. Curie saw the two racers leave at the same time, run at the same constant speed, $v$, and reach her at exactly the same time. She called this race a tie as well. Einstein was furious, claiming that Bohr had cheated.*

*Let E be a reference frame moving at speed $v$ in the same direction as Einstein*[1].

[1] The frame E is assumed to always have been moving with this velocity, even before the race has started.

**2a)** *In reference frame E, show that Bohr starts the race earlier than Einstein, and give the expression for how much.*

In the lab (i.e., Curie's) frame, let Einstein start at position $x = -D$ at $t = 0$ and move in the positive $x$ direction. Bohr starts at $x = +D$ and at the same lab frame time $t = 0$. So we have two space-time events, $x^\mu_E = (0, -D)$ and $x^\mu_B = (0, D)$

Transforming to frame E (the primed frame) with the Lorentz transformations in the matrix representation, we have the space-time location of Einstein's start as

$$\begin{pmatrix} ct'_E \\ x'_E \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ -D \end{pmatrix} = \begin{pmatrix} \gamma\beta D \\ -\gamma D \end{pmatrix} \tag{13}$$

Similarly we find that the space-time location of Bohr leaving is $x'^\mu_B = (-\gamma\beta D, \gamma D)$. The zeroth component gives the time of each person's start in frame E. So Einstein leaves later than $t' = 0$ and Bohr leaves earlier. The total headstart of Bohr is

$$\Delta t' = -2\gamma\beta D \tag{14}$$

**2b)** *in reference frame E, what is the outcome of the race?*

It is also tie. In the lab frame, both Einstein's and Bohr's finish are at the same space-time event $t_E = t_B = D/v$ and $x_E = x_B = 0$. So even if you transform to another frame, the two finishes will be at the same time. Einstein can't deny that both he and Bohr reach Curie at the same time, however he claims that Bohr cheated by leaving earlier.

**2c)** *Bohr also claimed (more respectfully) that Einstein cheated. Draw a space-time diagram (or diagrams) that illustrates how it is that both Einstein and Bohr can think the other person cheated by starting earlier.*

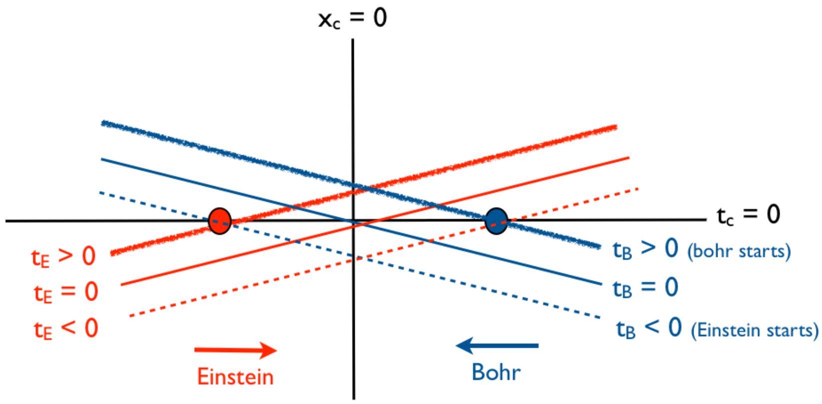


Figure 1: Both Einstein and Bohr start at $t = 0$ in Curie's frame (black axes). Einstein is assumed to move in the positive x-direction, and starts from the space-time point shown as a red circle. His lines of simultaneity (shown in red) are therefore skewed up. It is seen that in this frame, Bohr's start (the blue circle) occurs before Einsteins. At the same time, Bohr moves in the negative x-direction and his lines of simultaneity are skewed in the opposite way. It is seen that in this frame, Einstein leaves first.

**Problem 3: Einstein Charges (30 pts)**

UPSET BY THE RULING *of their second race, Einstein charged at Bohr at a speed $v_E$ very close to the speed of light. Einstein pulled out a laser gun that fires photons of energy $E_p$ with respect to the gun. He aimed and shot a photon straight ahead at Bohr.*

*Ever quick on his feet, Bohr pulled out a mirror and reflected the photon directly back at Einstein. The impulse of the photon flung Bohr backward. Einstein's eyes widened...*

**3a)** *Assuming the photon bounced off of Bohr's mirror elastically (i.e. the photon in the lab frame did not lose any energy in the bounce) solve for the speed at which Bohr is flung backward in terms of $E_p$, $v_E$, and Bohr's mass $m_B$.*

First we note that, since Einstein is moving, the energy and momentum of the photon in the lab frame is different than that in Einstein's frame (where $E_p$ is given). In Einstein's frame, the photon moves in the positive $x$ direction and it's momentum four vector is $p^\mu = (E_p/c, E_p/c)$. To transform this to the lab frame, we must note that, relative to Einstein the lab is moving in the *negative x* direction, so the velocity has a negative sign $\beta \to -\beta$. Applying the transformation matrix

$$\begin{pmatrix} E_{\text{lab}}/c \\ E_{\text{lab}}/c \end{pmatrix} = \begin{pmatrix} \gamma_E & \beta_E\gamma_E \\ \beta_E\gamma_E & \gamma_E \end{pmatrix} \begin{pmatrix} E_p/c \\ E_p/c \end{pmatrix} = \begin{pmatrix} \gamma_E E_p(1+\beta_E)/c \\ \gamma_E E_p(1+\beta_E)/c \end{pmatrix} \tag{15}$$

We use the subscript $E$ to clarify that the velocities in these expressions are Einstein's velocity $v_E$. Thus, the photon momentum in the lab frame is

$$p_l = \gamma_E E_p (1 + \beta_E)/c \tag{16}$$

You can check that the plus sign in $(1 + \beta_E)$ makes physical sense. If Einstein is charging at you, the energy of the photon he sends will be blue shifted, i.e., it will have it's energy and momentum boosted *up*.

To calculate Bohr's final velocity, we use conservation of momentum and the fact that the momentum of the photon changes sign on the reflection, but does not change magnitude, so the initial and final values are related by $p_f = -p_i$. Then the relativistic conservation of momentum equation is

$$p_l = -p_l + \gamma_B m_B v_B \tag{17}$$

or

$$\gamma_B v_B = 2p_l/m_B \tag{18}$$

This is the essential physics. To solve for $v_B$ we just square both sides and do some algebra

$$\begin{aligned}
\frac{v_B^2}{1 - v_B^2/c^2} &= 4p_l^2/m_B^2 && (19)\\
v_B^2 &= 4p_l^2/m_B^2(1 - v_B^2/c^2) && (20)\\
v_B^2(1 + 4p_l^2/m_B^2c^2) &= 4p_l^2/m_B^2 && (21)\\
v_B^2 &= 4p_l^2/m_B^2(1 + 4p_l^2/m_B^2c^2)^{-1} && (22)\\
v_B^2 &= c^2(m_B^2c^2/4p_l^2 + 1)^{-1} && (23)\\
v_B &= c[m_B^2c^2/4p_l^2 + 1]^{-1/2} && (24)
\end{aligned}$$

Since $p_l$ is given above, the problem is basically done. Optionally, we can plug in $p_l$ explicitly, and manipulate to a reasonably nice form

$$v_B = c\left[\left(\frac{m_B c^2}{\gamma_E E_p}\right)^2 \frac{1}{4(1 + \beta_E)^2} + 1\right]^{-1/2} \tag{25}$$

Note that the units are correct, and $v_B < c$, as it should be. If you want to approximate $\beta_E \approx 1$, this is

$$v_B \approx c\left[\left(\frac{m_B c^2}{4\gamma_E E_p}\right)^2 + 1\right]^{-1/2} \tag{26}$$

**Problem 4: Charged Medium (25 pts)**

IMAGINE A REGION OF SPACE *that is uniformly filled with positive charges, such that in the lab frame there is a charge density* $\rho$*. The charges are all moving in the positive x-direction with speed v, and hence the current density in the lab frame is* $\vec{J} = \rho v \hat{x}$*. It turns out that* $\rho$ *and J form a 4-vector* $J^\mu = (c\rho, \vec{J})$

**4a)** *Write down a quantity involving* $\rho$ *and* $\vec{J}$ *that is a Lorentz invariant.*

The inner product, or norm, of a four vector is always a Lorentz invariant. Therefore

$$-\rho^2 c^2 + \vec{J} \cdot \vec{J} \tag{27}$$

is a Lorentz invariant. Or if you are using the other metric convention, one can also use $\rho^2 c^2 - J^2$ just as well.

**4b)** *If you fly in the positive x-direction with speed v, such that the charges appear at rest, what is the charge density in this frame?*

This four vector transforms just like any other; to get values in the moving (primed) frame we apply the Lorentz transform matrix. We will only include the $x$ component of $\vec{J}$ since the $y, z$ components are zero.

$$\begin{pmatrix} c\rho' \\ J' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} c\rho \\ J \end{pmatrix} = \begin{pmatrix} \gamma c\rho - \gamma\beta J \\ \gamma J - \gamma\beta c\rho \end{pmatrix} \tag{28}$$

So the charge density in the primed frame is just the zeroth component

$$c\rho' = \gamma(c\rho - \beta J) \tag{29}$$

Using the given fact that $J = \rho v = c\rho\beta$ we see

$$c\rho' = \gamma(c\rho - c\rho\beta^2) = \gamma c\rho(1 - \beta^2) = \gamma c\rho\gamma^{-2} \tag{30}$$

and so

$$\rho' = \rho\gamma^{-1} \tag{31}$$

Although it was not asked for, note that the current density in the primed frame is

$$J' = \gamma J - \gamma\beta c\rho = \gamma\beta c\rho - \gamma\beta c\rho = 0 \tag{32}$$

As it should be, since in this frame the charges are at rest.

**4c)** *Give a simple explanation why the charge density is different by the factor you found in part b).*

Let $l$ be the separation between charges in the x-direction in the frame in which they are rest. The measured length of something is smaller (Lorentz contracted) when it is observed in a frame where it is moving (i.e., the proper length is always maximal); so in the lab frame, the distance between the charges is smaller, $l_{\text{lab}} = l/\gamma$. Thus the density in the lab frame should be greater by a factor of $\gamma$, so $\rho = \rho' \gamma$, as found in 4b).

**4d)** *If you fly in the negative x-direction with speed v, what is the current density, $\vec{J}'$ in this frame?*

This problem is like part 4b) but we transform to a frame moving in the negative $x$ direction, so $\beta \to -\beta$

$$\begin{pmatrix} c\rho' \\ J' \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} c\rho \\ J \end{pmatrix} = \begin{pmatrix} \gamma c\rho + \gamma\beta J \\ \gamma J + \gamma\beta c\rho \end{pmatrix} \tag{33}$$

So we find

$$J' = \gamma(J + \beta c\rho) \tag{34}$$

Using the fact that $J = c\beta\rho$ this can be written

$$J' = 2\gamma v\rho = 2\gamma J \tag{35}$$

So the current not only doubles, it is increases by a factor $\gamma$.