

Problem # 1:

We need the possible wavelengths of standing modes on the string.
We can either do this by inspection or via boundary conditions.

Inspection: Node at left end, antinode at right end.

can fit $\frac{\lambda}{4}$ on string, and can add any number of half-wavelengths.

$$L = \frac{\lambda}{4} + \frac{n\lambda}{2} \Rightarrow \lambda = \frac{4L}{2n+1}, n \geq 0$$

Boundary Conditions: The amplitude of the standing wave goes as $A \sin(kx + \phi)$. Let us label the left end $x=0$, and the right end $x=L$.

$$\text{At } x=0, A(x) = A \sin(kx + \phi) = A(0) = A \sin \phi = 0$$

$$\text{so } \sin \phi = 0 \Rightarrow \phi = 0$$

$$A(x) = A \sin kx$$

$$\text{At } x=L, A(L) = A \sin kL = \pm A$$

$$\sin kL = \pm 1$$

$$kL = \frac{\pi}{2} + n\pi$$

$$k = \left(\frac{\pi}{2} + n\pi\right) \frac{1}{L}$$

$$\frac{2\pi}{\lambda} = \left(\frac{\pi}{2} + n\pi\right) \frac{1}{L}$$

$$\lambda = \frac{4L}{2n+1}$$

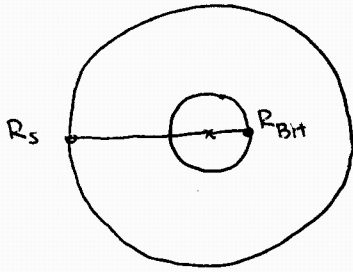
$$v = \lambda f \Rightarrow f = \frac{v}{\lambda}, \quad v = \sqrt{\frac{F_T}{\mu}}, \quad F_T = mg \text{ (mass does not move)}$$

$\mu = m/L$ (uniform string)

$$v = \sqrt{gL} \Rightarrow f = \frac{2n+1}{4L} \sqrt{gL}$$

$$f = \frac{2n+1}{4} \sqrt{g/L}$$

2.



Take the center of mass as the origin.

$$\frac{M_{BH} R_{BH} - M_s R_s}{M_{BH} + M_s} = 0$$

$$\Rightarrow M_s R_s = M_{BH} R_{BH} \quad (1)$$

- Calculate the orbital period of the star:

$$F_g = F_{cent} \quad (\text{circular motion})$$

$$\frac{G M_s M_{BH}}{(R_s + R_{BH})^2} = \frac{M_s v_s^2}{R_s} \quad \text{by (1), } R_{BH} = \frac{M_s}{M_{BH}} R_s$$

$$\Rightarrow \frac{G M_{BH}}{(R_s + \frac{M_s}{M_{BH}} R_s)^2} = \frac{G M_{BH}}{R_s^2 (1 + \frac{M_s}{M_{BH}})^2} = \frac{v_s^2}{R_s}$$

$$R_s = \frac{G M_{BH}}{v_s^2 (1 + \frac{M_s}{M_{BH}})^2} \quad (= 1.88 \cdot 10^9 \text{ m})$$

$$T_s = \frac{2\pi R_s}{v_s} = \frac{2\pi G M_{BH}}{v_s^3 (1 + \frac{M_s}{M_{BH}})^2} = 2.69 \cdot 10^4 \text{ s}$$

- Calculate the orbital velocity of the black hole
 - you could start with $F_g = F_{cent}$ and derive the result in that way, but it's simpler to notice that the star and the black hole must have the same orbital period if they are to remain in orbit

$$T_s = T_{BH}$$

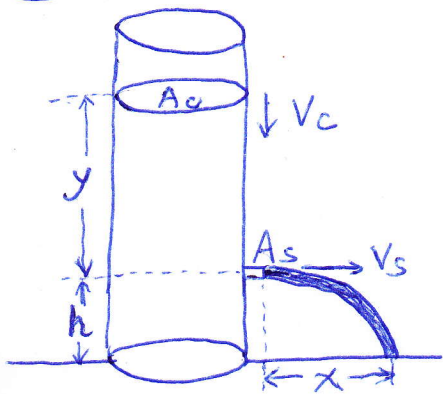
$$\frac{2\pi R_s}{v_s} = \frac{2\pi R_{BH}}{v_{BH}} \Rightarrow \frac{R_s}{v_s} = \frac{M_s R_s}{M_{BH} v_{BH}} \Rightarrow v_{BH} = \frac{M_s v_s}{M_{BH}}$$

$$= 77.6 \text{ km/s}$$

- Total Angular Momentum:

$$\begin{aligned} L &= L_s + L_{BH} = R_s M_s v_s + R_{BH} M_{BH} v_{BH} \\ &= R_s M_s v_s + R_s M_s v_{BH} \\ &= 1.30 \cdot 10^{45} \text{ kg m}^2/\text{s} \end{aligned}$$

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$$A_c V_c = A_s V_s \quad (1)$$

$$\frac{1}{2} \rho V_c^2 + \rho g y + P_{atm} = \frac{1}{2} \rho V_s^2 + P_{atm} \quad (2)$$

From (1), $V_s = \frac{A_c}{A_s} V_c$

From (2), $\frac{1}{2} \rho V_c^2 + \rho g y = \frac{1}{2} \rho \frac{A_c^2}{A_s^2} V_c^2$,

so $V_c = \sqrt{\frac{2gy}{\frac{A_c^2}{A_s^2} - 1}}$, $V_s = \sqrt{\frac{2gy}{1 - A_s^2/A_c^2}}$

The time it takes for the water to fall to the ground is $t = \sqrt{2h/g}$, so the distance is

$$x = V_s t = \sqrt{\frac{4hy}{1 - A_s^2/A_c^2}}$$

$$V = -\frac{dx}{dt} = \left(\frac{4h}{1 - A_s^2/A_c^2}\right)^{\frac{1}{2}} \cdot \frac{1}{2\sqrt{y}} \cdot \left(-\frac{dy}{dt}\right)$$

$$= \left(\frac{4h}{1 - A_s^2/A_c^2}\right)^{\frac{1}{2}} \cdot \frac{1}{2\sqrt{y}} \cdot \sqrt{\frac{2gy}{A_c^2/A_s^2 - 1}}$$

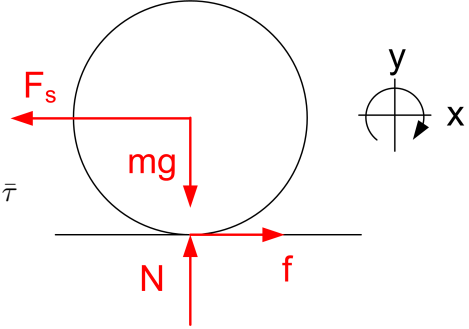
$$V = \frac{A_s/A_c}{(1 - A_s^2/A_c^2)} \sqrt{2gh}$$

Final Exam Solution: Problem 4

Physics 7A, UC Berkeley, Fall 2010, Prof. A. Speliotopoulos
Grader: Aaron Alpert

Force Considerations. The normal force is $N = mg$, and therefore, the maximum frictional force is $f_{max} = \mu_s N = \mu_s mg$. The rolling without slipping condition¹ is $\bar{v} = r\omega$, which implies $\bar{a} = r\alpha$. (Simply take the time derivative to accomplish this.) That implies

$$\frac{1}{m} \sum F = \frac{r}{I} \sum \bar{\tau} \quad \rightarrow \quad \frac{1}{m} \sum F = \frac{r}{\frac{2}{5}mr^2} \sum \bar{\tau} = \frac{5}{2mr} \sum \bar{\tau} \quad (1)$$



The two relevant forces are friction and the spring force. When the friction is maximized, so is the spring force. At the maximal condition,

$$\frac{f_{max} - F_{s,max}}{m} = \frac{5}{2mr} (-rf_{max}) \quad \rightarrow \quad F_{s,max} = \frac{7}{2}f_{max} \quad (2)$$

As previously stated, the maximum frictional force is $\mu_s mg$, which leads to $F_{s,max} = -\frac{7}{2}\mu_s mg$.

Energy Considerations. Using Hooke's Law, we can find the maximum displacement (denoted as δ) which corresponds to the maximum force.

$$F_s = -kx \quad \rightarrow \quad F_{s,max} = -\frac{7}{2}\mu_s mg = -k\delta \quad \rightarrow \quad \delta = \frac{7\mu_s mg}{2k} \quad (3)$$

The total energy stored in a spring stretched δ is

$$E_s = \frac{1}{2}k\delta^2 = \frac{1}{2}k \left(\frac{7\mu_s mg}{2k} \right)^2 = \frac{49\mu^2 m^2 g^2}{8k} = \frac{49(.8)^2 (2)^2 (9.8)^2}{8(100)} = 15.06 \text{ J} \quad (4)$$

At the ball passes the equilibrium point, it no longer will have any potential energy, just kinetic energy from translation and rotation.

$$E_{eq} = \frac{1}{2}m\bar{v}_{max}^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}m\bar{v}_{max}^2 + \frac{1}{2} \frac{2}{5}mr^2 \left(\frac{\bar{v}_{max}}{r} \right)^2 = \frac{7}{10}m\bar{v}_{max}^2 \quad (5)$$

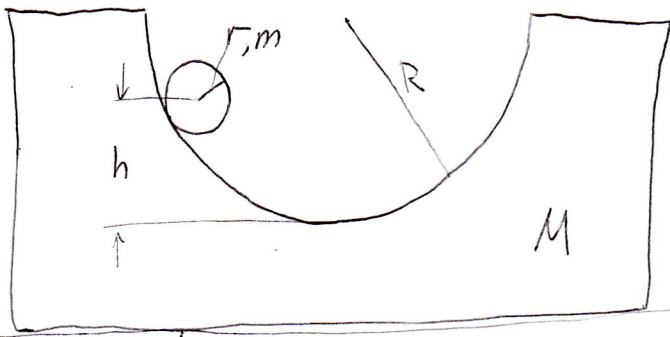
Using the conservation of energy,

$$E_{eq} = E_s \quad \rightarrow \quad \frac{7}{10}(2)\bar{v}_{max}^2 = 15.06 \text{ J} \quad \rightarrow \quad \bar{v} = \boxed{3.28 \frac{m}{s}} \quad (6)$$

Grading Rubric. 8 points for the force considerations (broken down into +2 for the rolling without slipping condition; +2 for $F = ma$; +2 for $\tau = I\alpha$; +2 for finding the max spring force). 12 points for the energy considerations (broken down into +2 for the maximum displacement; +4 for the energy in the spring; +4 for the kinetic and rotational energy; +2 for energy conservation).

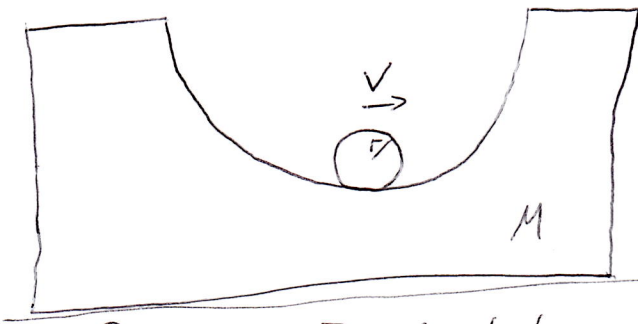
¹I am using the overbar, as in \bar{v} , to indicate that the quantity refers to the center of mass. Also, I had to carefully choose that clockwise rotation was positive, or the condition would be $\bar{v} = -r\omega$.

Physics 7A, Section 1
 (Speliotopoulos)
 Final Exam, Fall 2010
 Problem 5



Initial state

no friction



Final state

1) Momentum conservation: $0 = mv + Mu$

2) Energy conservation: $mg(h-r) = E_0 = E_f = \frac{1}{2}mv^2 + \frac{1}{2}Mu^2 + \frac{1}{2}I\omega^2$
 $I = \frac{2}{5}mr^2$

3) Rolling without slipping: $\omega r = -u + v$

$$\begin{aligned}
 mg(h-r) &= \frac{1}{2}mv^2 + \frac{1}{2}Mu^2 + \frac{1}{5}mr^2\omega^2 \\
 &= \frac{1}{2}mv^2 + \frac{1}{2}Mu^2 + \frac{1}{5}m(-u+v)^2 \\
 &= \frac{1}{2}mv^2 + \frac{1}{2}\frac{(Mu)^2}{M} + \frac{1}{5}m\left(+\frac{m}{M}v+v\right)^2 \\
 &= \frac{1}{2}mv^2 + \frac{1}{2}mv^2 \cdot \frac{m}{M} + \frac{1}{5}mv^2\left(1+\frac{m}{M}\right)^2 \\
 &= mv^2\left(\frac{1}{2} + \frac{1}{2}\frac{m}{M} + \frac{1}{5}\left(1+\frac{m}{M}\right)^2\right)
 \end{aligned}$$

$$v^2 = \frac{g(h-r)}{\frac{1}{2} + \frac{1}{2}\frac{m}{M} + \frac{1}{5}\left(1+\frac{m}{M}\right)^2} = \frac{10g(h-r)}{\left(\frac{m}{M}+1\right)\left(2\frac{m}{M}+7\right)} \quad v = \sqrt{\frac{10g(h-r)}{\left(\frac{m}{M}+1\right)\left(2\frac{m}{M}+7\right)}}$$

$$u = -\frac{mv}{M} = -\frac{m}{M} \sqrt{\frac{10g(h-r)}{\left(\frac{m}{M}+1\right)\left(2\frac{m}{M}+7\right)}}$$

Problem 6

December 17, 2010

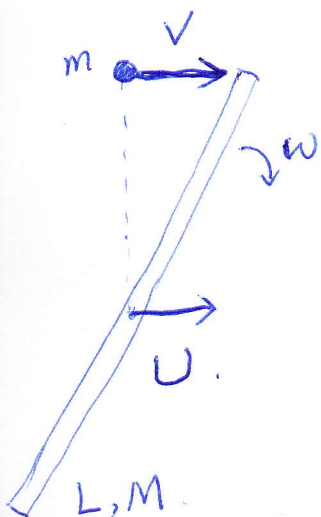
$$\begin{aligned}
 N_{hoop} &= m\omega^2 R \\
 N_{table} &= mg \\
 Fr_{hoop} &= \mu m\omega^2 R \\
 Fr_{table} &= \mu mg \\
 mr^2\alpha &= \tau_{net} = -R Fr_{hoop} - R Fr_{table} \\
 mr^2\alpha &= -\mu m(R^2\omega^2 + gR) \\
 \alpha &= -\mu\left(\omega^2 + \frac{g}{R}\right) \\
 \frac{d\omega}{dt} &= -\mu\left(\omega^2 + \frac{g}{R}\right) \\
 \frac{d\omega}{\omega^2 + \frac{g}{R}} &= -\mu \\
 \int_{\omega_0}^{\omega(t)} \frac{d\omega}{\omega^2 + \frac{g}{R}} &= \int_0^t -\mu dt \\
 \sqrt{\frac{R}{g}}\left(\tan^{-1}\left(\omega\sqrt{\frac{R}{g}}\right) - \tan^{-1}\left(\omega_0\sqrt{\frac{R}{g}}\right)\right) &= -\mu t \\
 \sqrt{\frac{R}{g}}\left(\tan^{-1}\left(\omega\sqrt{\frac{R}{g}}\right) - \frac{\pi}{4}\right) &= -\mu t \\
 \tan^{-1}\left(\omega\sqrt{\frac{R}{g}}\right) &= -\mu t\sqrt{\frac{g}{R}} + \frac{\pi}{4} \\
 \omega\sqrt{\frac{R}{g}} &= \tan\left(\frac{\pi}{4} - \mu t\sqrt{\frac{g}{R}}\right) \\
 \omega &= \sqrt{\frac{g}{R}}\tan\left(\frac{\pi}{4} - \mu t\sqrt{\frac{g}{R}}\right)
 \end{aligned}$$

$$\omega(T) = 0$$

$$\frac{\pi}{4} = \mu T \sqrt{\frac{g}{R}}$$

$$T = \frac{\pi}{4\mu} \sqrt{\frac{R}{g}}$$

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The only way to get a second collision is to assume $U = v$. (*)

(1) momentum: $mv_0 = mv + M U$

(2) energy: $\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}M U^2 + \frac{1}{2}\left(\frac{1}{12}ML^2\right)\omega^2$

(3) angular momentum: $\frac{mv_0 L}{2} = \frac{mvL}{2} + \left(\frac{1}{12}ML^2\right)\omega$

From (*) and (1), we find

(4) $U = v = \frac{mv_0}{m+M}$

From (3) and (4), we find

(5) $\omega = \frac{1}{\left(\frac{1}{12}ML^2\right)} \frac{mL}{2} (v_0 - v) = \frac{1}{\left(\frac{1}{12}ML^2\right)} \frac{mL}{2} \left(\frac{M}{m+M}\right) v_0$

substitute (4) and (5) into (2), we have

$$\frac{1}{2}mv_0^2 = \frac{1}{2}m \left(\frac{m}{m+M}\right)^2 v_0^2 + \frac{1}{2}M \left(\frac{m}{m+M}\right)^2 v_0^2 + \frac{1}{2} \left(\frac{1}{12}ML^2\right) \frac{\left(\frac{mL}{2}\right)^2}{\left(\frac{1}{12}ML^2\right)^2} \left(\frac{M}{m+M}\right)^2 v_0^2$$

which simplifies to

$$(m+M)^2 m = m^3 + Mm^2 + 3m^2M$$

$$\Rightarrow M^2 m = 2m^2 M$$

$$\Rightarrow M = 2m, \text{ or } \boxed{\frac{m}{M} = \frac{1}{2}}$$

Using this value of m/M in (4) and (5), we find

$$v = \frac{1}{3}v_0, \quad \omega = 6 \frac{m/M}{L} \left(\frac{1}{m/M+1}\right) v_0 = 2v_0/L$$

The mass will have travelled for half a period of the bar's rotational motion at speed v , so the distance is

$$d = v_0 \left(\frac{2\pi}{\omega}\right) \cdot \frac{1}{2} = \frac{1}{3}v_0 \left(\frac{2\pi}{2v_0/L}\right) \cdot \frac{1}{2} = \boxed{\frac{1}{6}\pi L}$$