Midterm 1 solutions

Problem 1:

Part a)

Let $x[n] = \delta[n]$, then we can iterate to solve for $h[n] = \delta[n] + \delta[n-1] + 0.25h[n-2]$. Starting with $y[n] = 0$ for $n \leq 1$, we obtain the impulse response

$$\begin{aligned}
h[0] &= \delta[0] + \delta[-1] + 0.25h[-2] = 1 \\
h[1] &= \delta[1] + \delta[0] + 0.25h[-1] = 1 \\
h[2] &= \delta[2] + \delta[1] + 0.25h[0] = 0.25 \\
h[3] &= \delta[3] + \delta[2] + 0.25h[1] = 0.25 \\
h[4] &= \delta[4] + \delta[3] + 0.25h[2] = 0.25^2 \\
h[5] &= \delta[5] + \delta[4] + 0.25h[3] = 0.25^2 \\
h[n] &= \begin{cases} 0.25^{n/2} & \text{for n even and n} \geq 0 \\ 0.25^{(n-1)/2} & \text{for n odd and n} \geq 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Or, using Fourier transform methods, we obtain the impulse response

$$\begin{aligned}
Y(e^{jw}) - 0.25e^{-2jw}Y(e^{jw}) &= X(e^{jw}) + e^{-jw}X(e^{-jw}) \\
H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} &= \frac{1 + e^{-jw}}{1 - 0.25e^{-2jw}} = -0.5\frac{1}{1 + 0.5e^{jw}} + 1.5\frac{1}{1 - 0.5e^{jw}} \\
h[n] &= -0.5(-0.5)^n u[n] + 1.5(0.5)^n u[n]
\end{aligned}$$

which is an equivalent expression.

Part b)

To test for stability, we check to see if the impulse response is absolutely summable

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} |h[k]| &= \sum_{k=-\infty}^{\infty} |-0.5(-0.5)^k u[k] + 1.5(0.5)^k u[k]| \\
&= \sum_{k=0}^{\infty} |-0.5(-0.5)^k + 1.5(0.5)^k| \\
&= -0.5\frac{1}{1+0.5} + 1.5\frac{1}{1-0.5} = \frac{8}{3} < \infty
\end{aligned}$$

Therefore, the LTI system is stable.

Problem 2:

The input is periodic with period $N = 5$, so the fundamental period of the output is also $N = 5$. First we compute the Fourier series coefficients of the input, a_k,

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi}{N}n} = \frac{1}{5} \qquad 0 \leq k \leq 4.$$

Then we find the frequency response of the system

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} = \frac{\sin(\frac{5\omega}{2})}{\sin(\frac{\omega}{2})}.$$

Since the system is LTI, the response to the signal $e^{j\omega_0 n}$ is $H(e^{j\omega_0})e^{j\omega_0 n}$. Therefore we have

$$b_k = H(e^{jk\frac{2\pi}{N}})\, a_k = \frac{1}{5} \frac{\sin(\pi k)}{\sin(\frac{\pi k}{5})} = \begin{cases} 1 & k = 0, \pm 5 \cdots \\ 0 & \text{o.w.} \end{cases},$$

where $\{b_k\}$ are the Fourier series coefficients of the output.

Problem 3:

Part a)

To determine the Fourier transform, notice that the continuous time signal is just a convolution of two square pulses

$$z(t) = \begin{cases} 1, & |t| < 1 \\ 0, & |t| > 1 \end{cases} \quad \rightarrow \quad Z(jw) = \frac{2\sin(\omega)}{\omega}$$

By using the convolution in time property

$$x(t) = z(t) * z(t) \quad \rightarrow \quad X(jw) = Z(jw)Z(jw) = \left(\frac{2\sin(\omega)}{\omega}\right)^2$$

Part b)

$$X(jw) = \begin{cases} -\omega e^{-2j\omega}, & -1 \leq \omega \leq 0 \\ \omega e^{-2j\omega}, & 0 < \omega \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

To find the continuous-time signal, we can evaluate the synthesis equation

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)| e^{j\angle X(j\omega)} e^{j\omega t} d\omega \\
&= \frac{1}{2\pi}\int_{-1}^{0} -\omega e^{-2j\omega} e^{j\omega t} d\omega + \frac{1}{2\pi}\int_{0}^{1} \omega e^{-2j\omega} e^{j\omega t} d\omega \\
&= \frac{1}{2\pi}\int_{-1}^{0} -\omega e^{j\omega t - 2j\omega} d\omega + \frac{1}{2\pi}\int_{0}^{1} \omega e^{j\omega t - 2j\omega} d\omega \\
&= \frac{1}{2\pi}\int_{0}^{1} \omega e^{-j\omega(t-2)} d\omega + \frac{1}{2\pi}\int_{0}^{1} \omega e^{j\omega(t-2)} d\omega \\
&= \frac{1}{2\pi}\int_{0}^{1} \omega (e^{-j\omega(t-2)} + e^{j\omega(t-2)}) d\omega \\
&= \frac{1}{\pi}\int_{0}^{1} \omega \cos\left(\omega(t-2)\right) d\omega, \text{ using integration by parts } \int u dv = uv - \int v du \\
&= \frac{1}{\pi}\left(\omega \frac{1}{t-2}\sin\left(\omega(t-2)\right) + \frac{1}{(t-2)^2}\cos\left(w(t-2)\right)\right)\Bigg|_0^1, \ t \neq 2 \\
&= \frac{1}{\pi}\left(\frac{1}{t-2}\sin(t-2) + \frac{1}{(t-2)^2}\cos(t-2) - \frac{1}{(t-2)^2}\right), \ t \neq 2
\end{aligned}
$$

When $t = 2$, the integral evaluates to

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{0}^{1} \omega (e^{-j\omega(t-2)} + e^{j\omega(t-2)}) d\omega, \ t = 2 \\
&= \frac{1}{\pi}\int_{0}^{1} \omega d\omega = \frac{1}{2\pi}, \ t = 2
\end{aligned}
$$

Problem 4:

Substituting $x[n] = e^{j\omega n}$ and using the fact that $y[n] = H(e^{j\omega})e^{j\omega n}$ (or equivalently by taking the Fourier transform from both sides), we obtain the frequency response

$$
H(e^{j\omega}) = \frac{b + e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}},
$$

with the norm

$$
|H(e^{j\omega})| = \frac{|b + e^{-j\omega}|}{|1 - \frac{1}{2}e^{-j\omega}|} = \frac{\sqrt{(b+\cos(\omega))^2 + \sin(\omega)^2}}{\sqrt{(1-\frac{1}{2}\cos(\omega))^2 + \frac{1}{4}\sin(\omega)^2}}.
$$

We want to have

$$
(b+\cos(\omega))^2 + \sin(\omega)^2 = (1 - \frac{1}{2}\cos(\omega))^2 + \frac{1}{4}\sin(\omega)^2.
$$

Doing a little bit of algebra, and using the fact that $\sin(\omega)^2 + \cos(\omega)^2 = 1$, one can see that $b = -\frac{1}{2}$ satisfies this equation.

Problem 5:

i) Group delay is just the slope of the phase response of the filter, such as

$$-\frac{d}{dw}\angle H(e^{j\omega}) \;=\; 1$$

which is equivalent to

$$\angle H(e^{j\omega}) \;=\; -w + Constant \quad (3\,points)$$

ii)

Since

$$h[n] \;=\; \frac{1}{2\pi}\int_{2\pi} H(e^{j\omega})e^{j\omega}dw$$

At n = 0, we have

$$h[0] = \frac{1}{2\pi}\int_{2\pi} H(e^{j\omega})dw \;=\; 1 \quad (4\,points)$$

iii) Since

$$\frac{1}{2\pi}\int_{2\pi} |H(e^{j\omega})|^2 dw \;=\; \sum_n |h[n]|^2 = 2$$
$$h^2[0] \,+\, h^2[1] \,+\, h^2[2] \;=\; 2$$
$$h^2[1] \,+\, h^2[2] \;=\; 1$$

Note: $\sum_n |h[n]|^2$ is different from $(\,h[0] \,+\, h[1] \,+\, h[2]\,)^2$ (4 points)

It should be clear now that we are looking for a solution taking the form such as

$$H(e^{j\omega}) \;=\; e^{-j\omega}\, A(e^{j\omega})$$

where the $A(e^{j\omega})$ is always a real number, and $e^{-j\omega}$ provides the linear phase response demanded in part i). From part ii) and iii), we have

$$H(e^{j\omega}) \;=\; 1 \,+\, h[1]e^{-j\omega} \,+\, h[2]e^{-2j\omega}$$
$$= e^{-j\omega}\,(e^{j\omega} \,+\, h[1] \,+\, h[2]e^{-j\omega})$$

Notice that if $h[1] = 0$ and $h[2] = 1$, the above equation can be simplified into $2cos(\omega)e^{-j\omega}$, which meet all three contraints. While with $h[1] = 0$ and $h[2] = -1$, it becomes $2sin(\omega)e^{-j\omega + j\frac{\pi}{2}}$.
Hence, the solution is NOT unique.
Also, it is okay if one recognized that it is analogous to the FIR filter example we covered in lectures (Feb23), and applied the conclusion. In that case, one needs to point out the desired FIR filter could either take the form of type I or III, therefore, the solution is again not unique.
(4 points)
For part b), the case that h[0]=h[2]=1 is shown.

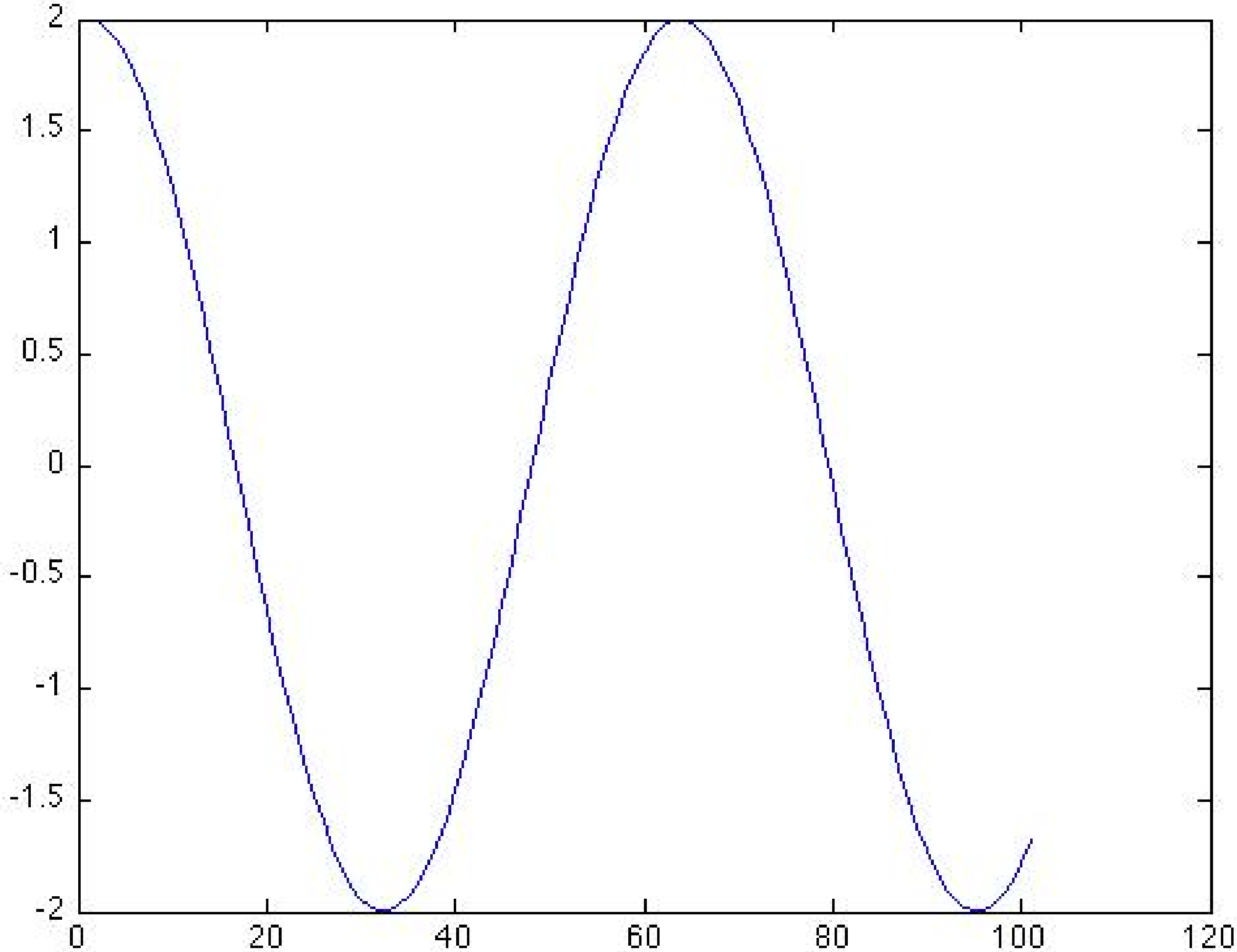

Figure 1: Some Figures

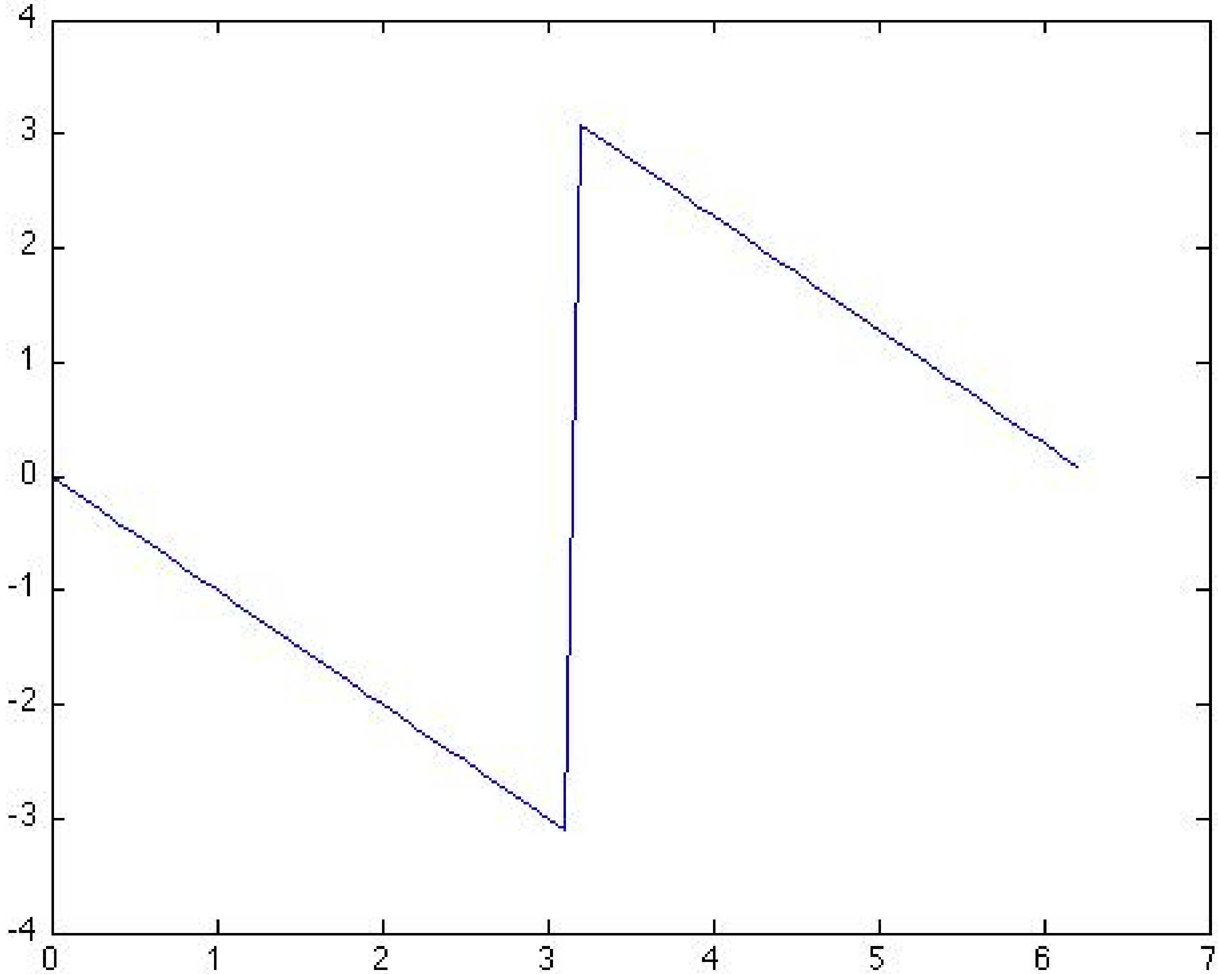

Figure 2: Some Figures