Key.

Problem 1 LTI Properties (24 pts)

[24 pts] Classify the following systems, with input $x(t)$ (or $x[n]$) and output $y(t)$ (or $y[n]$). In each column, write "yes", "no", or "?" if the property is not decidable with the given information. (+1 for correct, 0 for blank, -0.5 for incorrect).

System	Causal	Linear	Time-invariant	BIBO stable
a. $y(t) = x(t) * \sum_{n=0}^{\infty} \delta(t-4n)$	yes	yes	yes	no
b. $y(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t-4n)$	yes	yes	no	no
c. $y[n] = x[n] * 0.9^n u[n]$	yes	yes	yes	yes
d. $y[n] = \sum_{m=0}^{\infty} x[m] 0.9^m$ = constant	NO	Yes	no	Yes
d. $y(t) = x(t) * [\delta(t+1) + e^{-2t}u(t)]$	no	yes	yes	yes
e. $y(t) = x(t) * [\frac{d}{dt}\delta(t-1) + e^{-2t}u(t)]$	yes	yes	yes.	no

d) note $\Sigma\, 0.9^m = \frac{1}{1-0.9} = 10.$

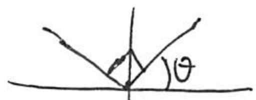
35' 1035

Problem 2 Short Answers (29 pts)

Answer each part independently. Note $\Pi(t) = u(t+\frac{1}{2}) - u(t-\frac{1}{2})$.

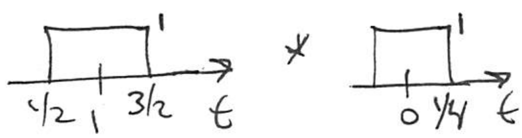
[3 pts] a. Evaluate $\delta(t+\frac{1}{4}) * \cos(2\pi t)u(t) =$ $\underline{\sin(2\pi t)\,u(t+1/4)}$

$\cos(2\pi(t+1/4))u(t+1/4) = \cos(2\pi t + \pi/2)u(t+1/4)$

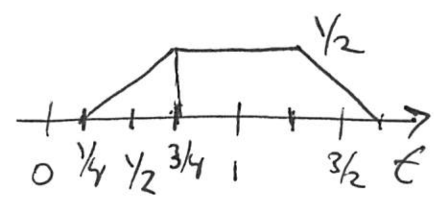


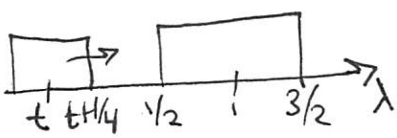
$\cos(\theta + \pi/2) = -\sin\theta$

[4 pts] b. Sketch $\Pi(t-1) * \Pi(2t)$



$\int h(\lambda)\,x(t-\lambda)\,d\lambda$





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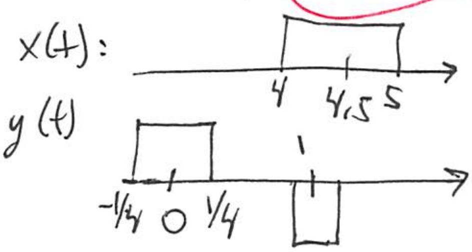
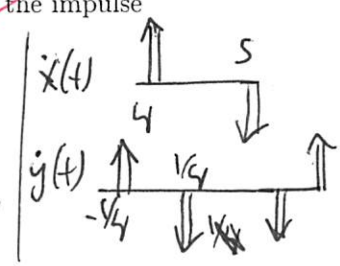
[4 pts] c. Given an LTI system with input $x(t) = \Pi(t-4.5)$ and output $y(t) = \Pi(2t)$ $-\Pi(2(t-1))$, find the impulse response of the system,

$h(t) =$ __________. (Hint: sketch $x(t)$ and $(y(t))$)

$h(t) = \delta(t+4\tfrac{1}{4}) - \delta(t+3\tfrac{3}{4})$





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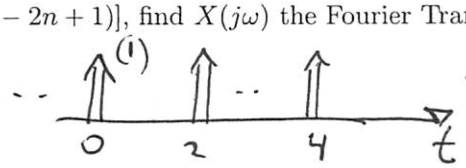
[4 pts] d. Given $x(t) = \sum_{n=-\infty}^{\infty}[\delta(t-2n) + \frac{1}{2}\delta(t-2n+1)]$, find $X(j\omega)$ the Fourier Transform of $x(t)$.

$X(j\omega) = \pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\pi)\left(1 + \frac{1}{2}(-1)^k\right)$



$= c(t)$

$\frac{\pi}{2} e^{-j\omega\cdot 1} \sum \delta(\omega - k\pi)$

$+ \frac{1}{2} c(t+1)$

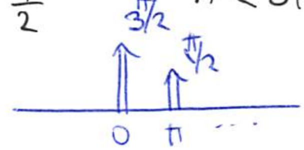
$C(j\omega) = \frac{2\pi}{2} \sum \delta(\omega - \frac{k2\pi}{2})$

$= \frac{\pi}{2} \sum e^{-jk\pi} \delta(\omega - k\pi)$

$\frac{1}{2} c(t+1) \Rightarrow \frac{1}{2}$

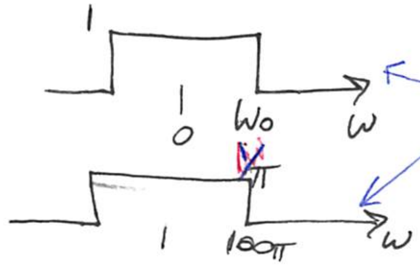
$= \pi \sum \delta(\omega - k\pi)$

$c(t-t_0) \rightarrow e^{-j\omega t_0} C(j\omega)$



$\frac{1}{\cancel{\pi^2}} = \frac{100\pi^2}{\cancel{\pi^2}}$ $\int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^2(j\omega)\,d\omega$

[4 pts] e. What is the energy in the time signal $\frac{\sin(100\pi t)}{t}$? $\frac{\pi^2 \cdot 200\pi}{2\pi}$



$$x(t) = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{tj\omega t}\,d\omega = \frac{e^{j\omega_0 t}}{2\pi(jt)}\Bigg|_{-\omega_0}^{\omega_0} = \frac{\sin \omega_0 t}{\pi t}$$

[6 pts] f. A system with input $x(t)$ and output $y(t)$ is described by the following differential equation: $\frac{d^2}{dt^2}y + 4\frac{d}{dt}y + y = \frac{d}{dt}x + 2x$.

Assuming zero initial conditions, find the impulse response for this system.

$h(t) =$ _________

$$Y(s)\,(s^2 + \cancel{4}\,2s + 1) = X(s)\,(s+2)$$

$$\frac{Y(s)}{X(s)} = \frac{s+2}{s^2+\cancel{4}\,2s+1} = \frac{1}{(s+1)^2} + \frac{1}{s+1}$$

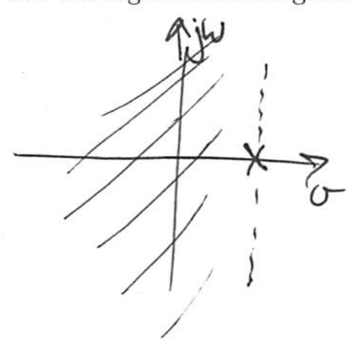
$$h(t) = \left[e^{-t} + te^{-t}\right]u(t).$$

[4 pts] g. Given the bilateral Laplace transform $X(s) = \frac{1}{s-2}$ and the region of convergence is to the left of the pole at $s = 2$, find the inverse Laplace transform,

$x(t) =$ $-e^{2t}u(-t)$

$$\int_{t=-\infty}^{0} e^{2t} e^{-st}\,dt$$

non-causal



$$= \frac{e^{(2-s)t}}{2-s}\Bigg|_{t=-\infty}^{0} = \frac{-1}{s-2}$$

$2 - \sigma > 0$

$\Rightarrow \sigma < 2$

10/7

$1120 \to 11{:}28$

Problem 3 Laplace Transform (14 pts)

[~~10~~ 6 pts] An LTI system has input $x(t)$, output $y(t)$ and transfer function

$$H(s) = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For input $x(t) = (1 + \cos\omega_n t)u(t)$, find the steady-state solution $y(t)$ for large t.

$y(t) = \frac{1}{\omega_n^2}\left(1 + \frac{\sin\omega_n t}{2\zeta}\right)$ $\quad X(s) = \frac{1}{s} + \frac{s}{s^2+\omega_n^2} \longrightarrow \boxed{H(s)} \longrightarrow Y_1(s) + Y_2(s)$

+2

+6

+6

unit step by FVT

$$\lim_{s\to 0} \frac{s}{s} H(s) = \frac{1}{\omega_n^2}$$

Steady state response

$$\cos\omega_n t\, u(t) \longrightarrow \frac{1}{2}\left(e^{+j\omega_n t} + e^{-j\omega_n t}\right)$$

$$H(j\omega_n) = \frac{1}{\underbrace{(j\omega_n)^2}_{-\omega_n^2} + 2\zeta\omega_n(j\omega_n) + \omega_n^2} = \frac{1}{j2\zeta\omega_n^2}$$

$$H(-j\omega_n) = \frac{-1}{j2\zeta\omega_n^2}, \quad y_2(t) = \frac{1}{4\zeta\omega_n^2 j}\left[\frac{e^{j\omega_n t}}{j} - e^{-j\omega_n t}\right]$$

$$= \frac{\sin(\omega_n t)}{2\zeta\omega_n^2}$$

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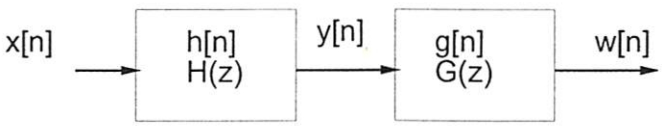
Problem 4. Z transform (29 pts)



[10 pts] a. Consider an LTI causal system with impulse response $h[n] = (2-(\frac{1}{2})^n)u[n]$. Find $g[n]$ such that $h[n]*g[n] = \delta[n]$.

$g[n] = \delta[n] - \frac{3}{2}\delta[n-1] + \frac{1}{2}\delta[n-2]$

$$H(z) = \frac{2}{1-z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{2-z^{-1}-1+z^{-1}}{(1-z^{-1})(1-\frac{1}{2}z^{-1})} = \frac{1}{(1-z^{-1})(1-\frac{1}{2}z^{-1})} = \frac{z^2}{(z-1)(z-\frac{1}{2})}$$

$$G(z) = \frac{(z-1)(z-\frac{1}{2})}{z^2} = \frac{z^2-\frac{3}{2}z+\frac{1}{2}}{z^2} = 1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}$$

+6 for G(z)

[3 pts] b. Show by direct calculation that the $g[n]$ from part a. above is the inverse of $h[n]$ from part a.

$g[n]*h[n] = h[n] - \frac{3}{2}h[n-1] + \frac{1}{2}h[n-2]$

+2 for writing terms out

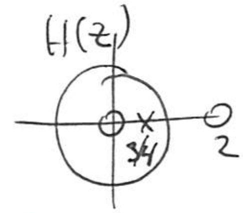
n	$h[n]$	$g[n]*h[n]$
0	1	$h[0]=1$
1	3/2	$h[1]-\frac{3}{2}h[0]=0$ ✓
2	7/4	$h[2]-\frac{3}{2}h[1]+\frac{1}{2}h[0]=0$ ✓
3	15/8	

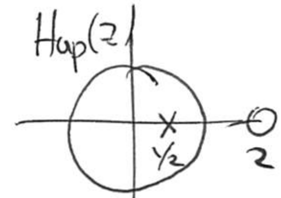
[4 pts] c. Show that $g[n]$ is the impulse response of a stable system.

G(z) has all poles in unit circle. Also $\sum_n |g[n]|^2$ is finite so g[n] is BIBO stable.

[12pts] d. Consider an LTI causal system with Z transform

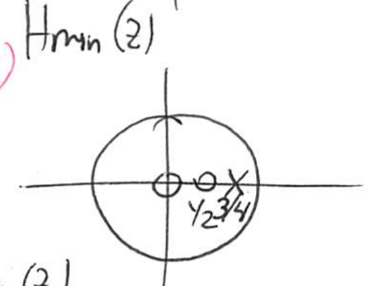
$$H(z) = \frac{z(z-2)}{z-3/4}$$





Find a **stable** $G(z)$ such that $|H(e^{j\Omega})G(e^{j\Omega})| = 1$ for all Ω.

$$G(z) = \frac{1}{H_{min}(z)} = \frac{1}{2}\frac{z-3/4}{z(z-\frac{1}{2})} = \frac{(1-\frac{3}{4}z^{-1})z^{-1}}{2(1-\frac{1}{2}z^{-1})} = \frac{(z-3/4)}{z(2z-1)}$$



$$\left|H(e^{j\Omega})G(e^{j\Omega})\right| = \left|\frac{e^{j\Omega}-2}{e^{j\Omega}-1/2}\right| = 1 ✓ \quad \frac{}{1/2} = 2$$

$$H(z) = H_{ap}(z)\cdot H_{min}(z) = \left(\frac{z-2}{z-\frac{1}{2}}\right)\left(\frac{z(z-\frac{1}{2})}{z-3/4}\right)$$

+2 if $\frac{1}{H(z)}$

+10 if within scale factor

+6 if using all pass

1148

Problem 5. Z Transform (24 pts)

A causal system with input $x[n]$ and output $y[n]$ is described by the difference equation:

$$y[n] + 0.3y[n-1] - 0.4y[n-2] = x[n] - x[n-1]$$

[12 pts] a. Find $Y(z)$ and $y[n]$ for $x[n] = 0$ (ZIR), with $y[-2] = \cancel{1}\ 4$ and $y[-1] = 2$.

$Y(z) = \dfrac{1}{1-0.5z^{-1}}$ $\qquad$ $y[n] = \left(\frac{1}{2}\right)^n u[n]$

$$Z\{y[n-1]\} = \sum_{n=0}^{\infty} y[n-1]z^{-n} = y[-1] + z^{-1}y[0] + \cdots = y[-1] + z^{-1}Y(z)$$

$$Z\{y[n-2]\} = \sum_{n=0}^{\infty} y[n-2]z^{-n} = y[-2] + z^{-1}y[-1] + z^{-2}y[0] + \cdots = y[-2] + z^{-1}y[-1] + z^{-2}Y(z)$$

$$Y(z) + 0.3(y[-1] + z^{-1}Y(z)) - 0.4(y[-2] + z^{-1}y[-1] + Y(z))$$
$$= X(z) - (x[-1] + z^{-1}X(z))$$

$$Y(z)(1 + 0.3z^{-1} - 0.4z^{-2}) = X(z)(1 - z^{-1}) - 0.3y[-1] + 0.4(y[-2] + z^{-1}y[-1]) - x[-1]$$

[12 pts] b. Find $Y(z)$ and $y[n]$ for $x[n] = u[n]$ (ZSR). $y[-2] = 0$ and $y[-1] = 0$.

$Y(z) = \dfrac{1}{\cancel{(1-z^{-1})}} \cdot \dfrac{\cancel{(1-z^{-1})}}{(1+0.8z^{-1})(1-0.5z^{-1})}$ $\qquad$ $y[n] = \frac{8}{13}\left(-\frac{4}{5}\right)^n u[n] + \frac{5}{13}\left(\frac{1}{2}\right)^n u[n].$

$$Y(z) = \frac{A}{1+0.8z^{-1}} + \frac{B}{1-0.5z^{-1}} = \frac{8/13}{1+4/5z^{-1}} + \frac{5/13}{1-\frac{1}{2}z^{-1}}$$

ZIR:

$$Y(z) = \frac{-0.3\overset{2}{y[-1]} + 0.4\overset{4}{y[-2]} + \overset{.4}{0.4}z^{-1}\overset{2}{y[-1]}}{(1+\underset{0.8}{\cancel{0.4}}z^{-1})(1-\underset{0.5}{\cancel{0.}}z^{-1})} = \frac{-.6 + 1.6 + 0.8z^{-1}}{(1+0.8z^{-1})(1-0.5z^{-1})} = \frac{1}{1-0.5z^{-1}}$$

0.6

0.3

Key.

Problem 6. Digital Filter (34 pts)

A continuous time causal LTI filter has transfer function

$$H(s) = 4\frac{s+1}{s+4} \qquad Y(s)(s+4) = 4(s+1)X(s)$$

[4 pts] a. Find the linear differential equation with constant coefficients with input $x(t)$ and output $y(t)$ which has transfer function $H(s)$ (assume zero initial conditions).

LDE: $\dot{y} + 4y = 4\dot{x} + 4x$

[4 pts] b. Using the backward difference approximation for the derivative (i.e. $\frac{dy}{dt} \approx \frac{y[n]-y[n-1]}{T}$),

$$2(y[n]-y[n-1]) + 4y[n] = 4\cdot x[n] + 4\cdot 2(x[n]-x[n-1])$$

with $T = \frac{1}{2}$, find the linear difference equation approximation with input $x[n]$ and output $y[n]$.

LDE: $6y[n] - 2y[n-1] = 12x[n] - 8x[n-1]$

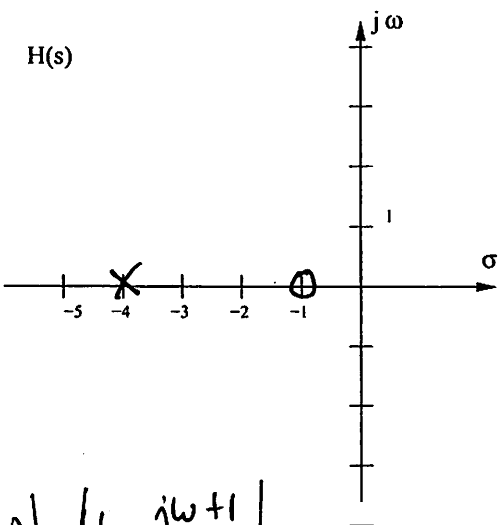
$$y[n] - \tfrac{1}{3}y[n-1] = \cancel{2x[n] - \ldots}\ 2\left[x[n] - \tfrac{2}{3}x[n-1]\right]$$

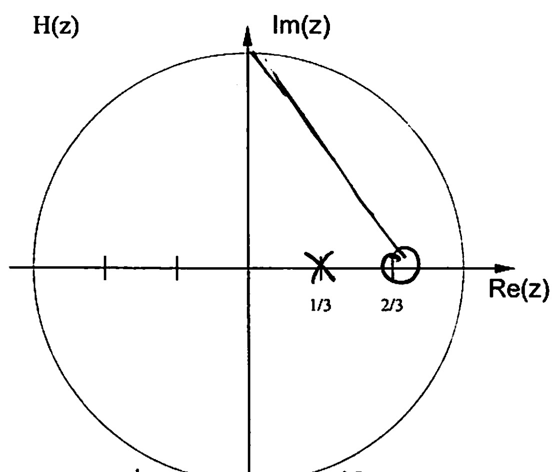
[4 pts] c. Assuming zero initial conditions, find the Z transform for the LDE in part b.

$$H(z) = \frac{2(1-\frac{2}{3}z^{-1})}{1-\frac{1}{3}z^{-1}}, \quad Y(z)\left(1-\tfrac{1}{3}z^{-1}\right) = 2X(z)\left(1-\tfrac{2}{3}z^{-1}\right)$$

$$= \frac{2(z-\frac{2}{3})}{z-\frac{1}{3}}$$

[4 pts] d. Draw pole-zero diagrams for $H(s)$ in the s-plane and $H(z)$ in the z-plane.

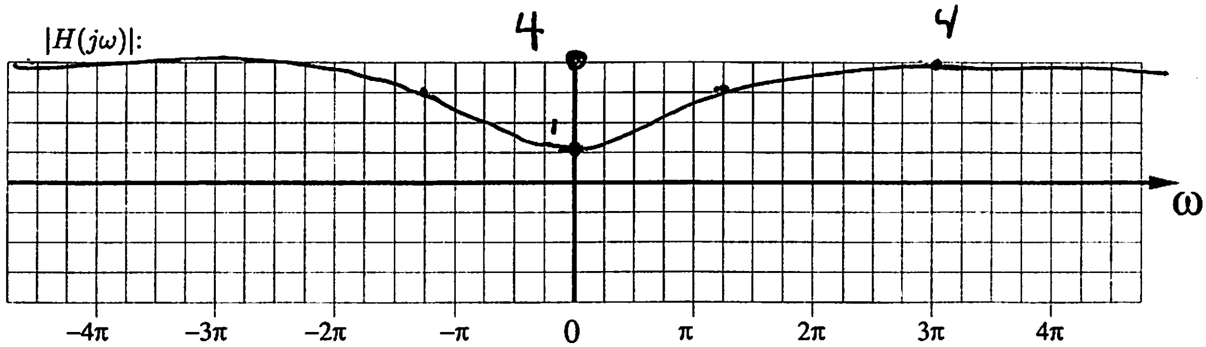




$$|H(j\omega)| = \left|4\,\frac{j\omega+1}{j\omega+4}\right| \qquad |H(e^{j\omega T})| = \left|\frac{2(e^{j\omega T}-\frac{2}{3})}{e^{j\omega T}-\frac{1}{3}}\right|$$

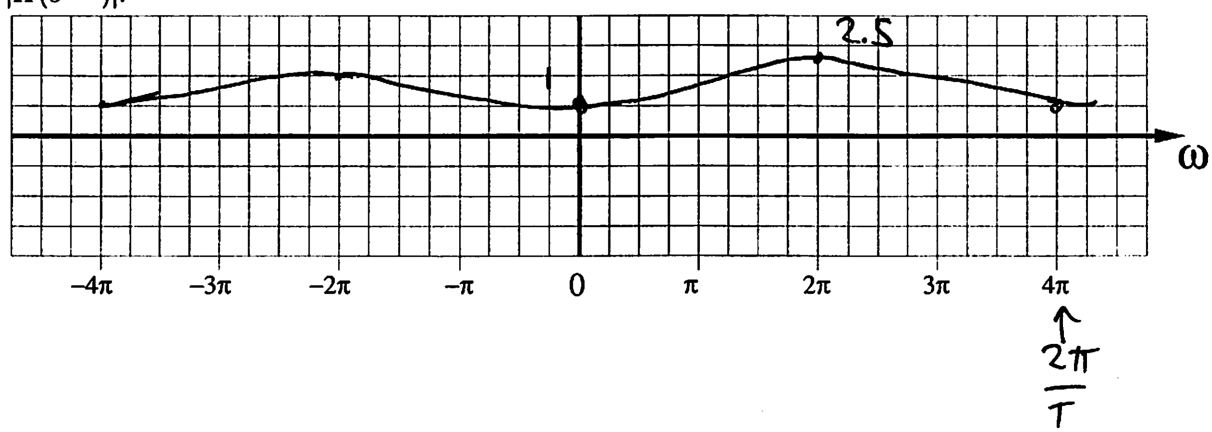
[8 pts] e. Sketch the magnitude of frequency response of the continuous time system, labelling key amplitudes.

$|H(j\omega)|$:



[8 pts] f. Sketch the magnitude of frequency response of the discrete time system, noting that $T = \frac{1}{2}$ sec., labelling key amplitudes.

$|H(e^{j\omega T})|$:



[2 pts] g. Briefly explain the reasons for differences between the magnitudes of the CT and DT frequency responses.

low frequencies approximately the same, but due to aliasing (periodic about 4π) does not match for $\omega > \sim \pi/2$.

Some possibly useful constants:

$\pi \approx 3.14$	$2\pi \approx 6.3$
$3\pi \approx 9.42$	$4\pi \approx 12.6$
$\sqrt{2} \approx 1.4$	$\sqrt{3} \approx 1.7$
$\sqrt{5} \approx 2.2$	$\sqrt{10} \approx 3.2$
$\sqrt{8} \approx 2.8$	$\sqrt{17} \approx 4.1$
$\sqrt{20} \approx 4.5$	$\sqrt{26} \approx 5.1$

w	$\lvert H(jw)\rvert$
0	1
1	$\frac{4\cdot\sqrt{2}}{\sqrt{17}} \approx 1.4$
2	$\frac{4\cdot\sqrt{5}}{\sqrt{20}} = \ldots \approx 2.2$
4	$\frac{4\cdot\sqrt{17}}{\sqrt{32}} = \frac{\sqrt{17}}{\sqrt{2}} \approx 3$
10	$\frac{4\cdot\sqrt{101}}{\sqrt{104}} \approx 4$

$|H(e^{j\omega})|$

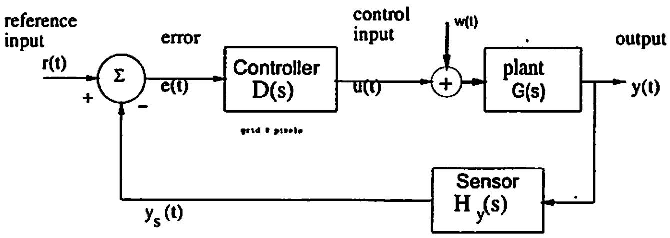
$\frac{2\cdot(1-2/3)}{1-1/3} = \frac{2\cdot 1/3}{2/3} = 1$

$|H(e^{j\pi})| = 2\cdot\frac{5/3}{4/3} = 2.5$ $\omega = 2\pi$

$|H(e^{j\pi/2})| = 2\cdot\frac{\sqrt{1+4/9}}{\sqrt{1+1/4}} \approx 2\cdot\frac{\sqrt{13}}{\sqrt{11}} \approx 2.1$

Key.

Problem 7. Control (24 pts)



$$E = R - HG(W + DE)$$

$$E(1 + HGD) = R - HGW$$

[3 pts] a. Find the transfer function $\frac{E(s)}{R(s)}$ in terms of D, G, H_y.

$\frac{E(s)}{R(s)} = \frac{1}{1 + HGD}$

[3 pts] b. Find the transfer function $\frac{E(s)}{W(s)}$ in terms of D, G, H_y.

$\frac{E(s)}{W(s)} = \frac{-HG}{1 + HGD}$

For the system above, let $D(s) = k_p$, $H_y(s) = \frac{s+1}{s}$, and $G(s) = \frac{1}{s^2+as+b}$.

$W(s) = \frac{1}{s}$

[10 pts] c. With $r(t) = 0$, determine trend of $e(t)$ as $t \to \infty$ with respect to a step disturbance input $w(t)$.

$e(t) \to$ $1/k$

$$\frac{E(s)}{W(s)} = \frac{s+1}{s(s^2+as+b) + K_p(s+1)}$$

$$\lim_{s \to 0} s \cdot \underbrace{\frac{1}{s}}_{W(s)} \cdot \frac{E(s)}{W(s)} = \frac{1}{K_p}.$$

[8 pts] d. With $w(t) = 0, H_y(s) = 1, D(s)G(s) = \frac{200}{(s+1)^2(s+10)^2}$,
determine ω_c for which $|D(j\omega_c)G(j\omega_c)| \approx 1$ and the approximate phase margin.
(Hint for small angles $\tan^{-1} 0.1 \approx 0.1$ rad $\approx 5.7°$.)

$\omega_c =$ 1

$$\angle D(j\omega)G(j\omega) = -2\angle(j\omega+1) - 2\angle(j\omega+10)$$
$$= -90° - 2\angle(j+10)$$
$$5.7°$$
$$= -101.4°$$

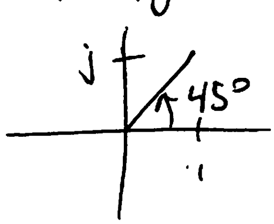
≈ 1.38 rad

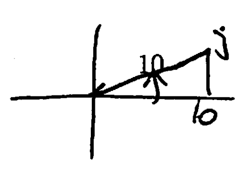
phase margin (specify rad or degrees) $\approx 79°$

$$\frac{200}{|j\omega+1|^2\,|j\omega+10|^2} = \frac{200}{(\omega^2+1)(\omega^2+100)} = 1, \quad (\omega^2+1)(\omega^2+100) = 200$$

assume $\omega^2 \ll 100$

$(\omega_c^2+1)\,100 \approx 200$

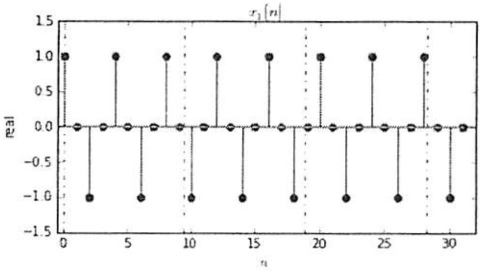




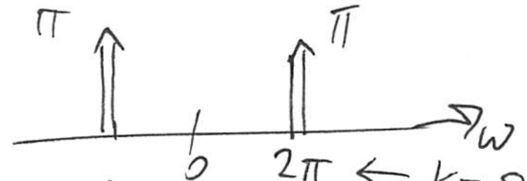
$\tan^{-1} \frac{1}{10} \approx 5.7°$ $\quad \omega_c \approx 1$

Problem 8. DFT problem or pole-zero match (22 pts)

[11 pts] a. Given $x_1[n] = \cos(2\pi \frac{n}{4})$ as shown:



$T_0 = 8$ sec



$\cos(2\pi t)$ $\quad t = nT = n/4$

$T = 1/4$ sec.

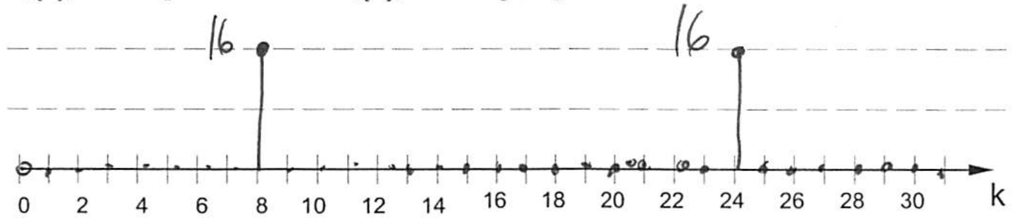
$\frac{2\pi}{T} = 8\pi$ ($k=32$)

bin spacing $\frac{8\pi}{32} = \pi/4$

sketch $X_1[k]$, the 32 point DFT of $x_3[n]$, labelling amplitudes.

amplitude −4

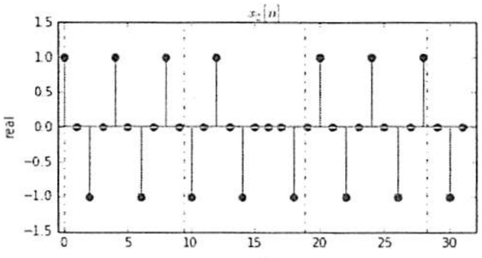
location −4



$X_1[k]$:

from DFT H.O. $X[k] = \frac{T_0}{2\pi}\,\text{area}\left[X'(\ )\right] = \frac{8}{2\pi}\cdot\pi\cdot\frac{1}{T} = 16$

[11 pts] b. Given $x_2[n] = \cos(2\pi \frac{n}{4}) - \delta[n-16]$ as shown:



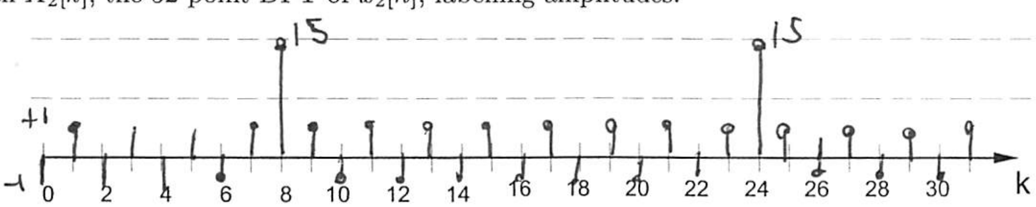
$X_2[k] = X_1[k] - \text{DFT}\{\delta[n-16]\}$

CFA

MISSING $X_1[k]$

−6

sketch $X_2[k]$, the 32 point DFT of $x_2[n]$, labelling amplitudes.



$X_2[k]$:

$$\text{DFT}\{\delta[n-16]\} = \sum_{n=0}^{31} \delta[n-16]\, e^{-j2\pi nk/32} = e^{-j2\pi\cdot 16k/32}$$

$$= e^{-j\pi k} = (-1)^k$$