University of California, Berkeley

Math 110, Spring 2013

Midterm 1

February 20, 2013

Name:

	Score
Exercise 1 (3 points)	
Exercise 2 (3 points)	
Exercise 3 (3 points)	
Readability (1 point)	
Total	

For each exercise, explain what you are doing, label each step you take with the axiom that allows you to take this step, and indicate which proof strategy you are using (contraposition, contradiction, etc.). The grading for these requirements will be negative; that is, we will remove points for failing to correctly justify a step taken. A "readability" point will be granted for the overall presentation of your solutions (i.e. readable writing, concise and precise sentences explaining what you are doing, etc.).

All the vectors spaces are over $\mathbb{R}$ or $\mathbb{C}$.

1. (3 points) Write out the definition of a subspace. Let U and W be subspaces of a vector space V. Prove that the intersection $U \cap W$ is a subspace of V.

Solution: .

Definition 0.1 *A subset U of a vector space V over $\mathbb{F}$ is a subspace if the following axioms are satisfied:*

(S1) $0 \in U$

(S2) if $u_1, u_2 \in U$*, then* $u_1 + u_2 \in U$

(S3) if $u \in U$ *and* $\lambda \in \mathbb{F}$*, then* $\lambda u \in U$.

One needs to check that axioms (S1), (S2), and (S3) are satisfied for the intersection $U \cap W$.

S1: Since U and W are subspaces, (S1) implies that $0 \in U$ and $0 \in W$. Thus, $0 \in U \cap W$.

S2: Let u_1 and u_2 be in $U \cap W$. This implies that $u_1, u_2 \in U$ and $u_1, u_2 \in W$. Since U and W are subspaces, (S2) implies that both $u_1 + u_2 \in U$ and $u_1 + u_2 \in W$. Thus $u_1 + u_2 \in U \cap W$.

S3: Let $u \in U \cap W$. Thus $u \in U$ and $u \in W$. Since U and W are subspaces, (S3) implies that $\lambda u \in U$ and $\lambda u \in W$ for all $\lambda \in \mathbb{F}$. Thus $\lambda u \in U \cap W$.

2. (3 points) Write out the definition of a linear map. Let V and W be vector spaces and let $T : V \to W$ be a linear map. Prove that $\operatorname{Null} T = \{0\}$ implies that T is injective.

CAUTION: (1) You can suppose that $\operatorname{Null} T$ is a subspace. (2) You will need to prove at some point that $v \neq w$ implies that $v - w \neq 0$.

Solution: .

Definition 0.2 *Let V and W be vector spaces over $\mathbb{F}$. A map $T : V \to W$ is a linear map if the following axioms are satisfied:*

(L1) $T(v_1 + v_2) = T(v_1) + T(v_2)$ for all $v_1, v_2 \in V$ (additivity)
(L2) $T(\lambda v) = \lambda T(v)$ for all $\lambda \in \mathbb{F}$ and $v \in V$ (homogeneity).

We want to show that

$$\operatorname{Null} T = \{0\} \implies T \text{ is injective.}$$

We proceed by contraposition; thus, we will prove

$$T \text{ is NOT injective} \implies \operatorname{Null} T \neq \{0\}.$$

Suppose T is not injective. Then there exists $v_1, v_2 \in V$ such that $v_1 \neq v_2$ and

$$T(v_1) \;=\; T(v_2). \tag{1}$$

Add the inverse of $T(v_1)$ on both sides of (1):

$$T(v_1) - T(v_1) = T(v_2) - T(v_1). \tag{2}$$

The left-hand side of (2) is equal to zero by the additive inverse axiom for vector spaces, while the right-hand side of (2) is equal to $T(v_2 - v_1)$ thanks to axioms (L1) and (L2).

Thus, we obtain that $0 = T(v_2 - v_1)$.

If we suppose the following

Lemma 0.1 *For $v_1, v_2 \in V$, then $v_1 \neq v_2$ implies that $v_1 - v_2 \neq 0$.*

then we can conclude that we have found a non-zero vector in $\operatorname{Null} T$ (namely $v_2 - v_1$). Hence, $\operatorname{Null} T \neq \{0\}$.

Proof of the lemma:

We proceed by contraposition and prove the following

$$v_1 - v_2 = 0 \implies v_1 = v_2.$$

Suppose that $0 = v_1 - v_2$. Then add v_2 on both sides:

$$\begin{aligned}
v_2 &= (v_1 - v_2) + v_2 \\
&= v_1 + (-v_2 + v_2) \quad \text{associativity axiom} \\
&= v_1 + 0 \quad \text{additive inverse axiom} \\
&= v_1 \quad \text{additive identity axiom}
\end{aligned}$$

3. (3 points) Write out the definition of linear independence. Let $(u_1, \dots, u_k)$ and $(w_1, \dots, w_l)$ be two lists of vectors in a vector space V. Consider the following subspaces of V:

$$U = \text{span}(u_1, \dots, u_k) \text{ and } W = \text{span}(w_1, \dots, w_l)$$

Prove that $U \cap W \neq \{0\}$ implies that the concatenated list $(u_1, \dots, u_k, w_1, \dots, w_l)$ is linearly *dependent*.

Solution: .

Definition 0.3 *A list of vectors $(v_1, \dots, v_k)$ is linearly independent if, whenever*

$$a_1 v_1 + \cdots + a_k v_k = 0$$

then $a_1 = \cdots = a_k = 0$.

We want to show

$$U \cap W \neq \{0\} \implies (u_1, \dots, u_k, w_1, \dots, w_l) \text{ is NOT linearly independent.}$$

Suppose that there is a non-zero $v \in U \cap W$. Since $(u_1, \dots, u_k)$ spans U and $v \neq 0$, there exists $a_1, \dots, a_k \in \mathbb{F}$ not all zero (otherwise we would have $v = 0$ which would contradict our assumption that $v \neq 0$) such that

$$v = a_1 u_1 + \cdots + a_k u_k. \tag{3}$$

For the same reason,

$$v = b_1 w_1 + \cdots + b_l w_l. \tag{4}$$

Equating (3) and (4), we obtain

$$a_1 u_1 + \cdots + a_k u_k = b_1 w_1 + \cdots + b_l w_l. \tag{5}$$

Adding the additive inverse of the right-hand side to both sides of (4) and using the additive inverse axiom, we obtain that

$$\begin{aligned} 0 &= a_1 u_1 + \cdots + a_k u_k - (b_1 w_1 + \cdots + b_l w_l) \\ &= a_1 u_1 + \cdots + a_k u_k + (-1)(b_1 w_1 + \cdots + b_l w_l) \quad \text{by Prop. 1.6} \\ &= a_1 u_1 + \cdots + a_k u_k + (-1) b_1 w_1 + \cdots + (-1) b_l w_l \quad \text{by the distributivity axiom} \\ &= a_1 u_1 + \cdots + a_k u_k + (-b_1) w_1 + \cdots + (-b_l) w_l \quad \text{by the associativity axiom} \end{aligned}$$

So we have found scalars $a_1, \dots, a_k, -b_1, \dots, -b_l \in \mathbb{F}$ not all zero such that

$$0 = a_1 u_1 + \cdots + a_k u_k + (-b_1) w_1 + \cdots + (-b_l) w_l,$$

which implies that the concatenated list $(u_1, \dots, u_k, w_1, \dots, w_l)$ is linearly dependent.