**Question 1.** (16 points) Choose the correct answers, worth 2 points each. No justification necessary. Incorrect answers carry a 2-point penalty, so random choices are not helpful. You may leave any question blank for 0 points.

T — Two $4 \times 6$ matrices with the same row space also have the same pivot positions.
*the reduced row echelon form is completely determined by the row space*

F — Every plane in $\mathbf{R}^3$ is a linear subspace.
*Only if it contains the origin*

F — Elementary row operations do not change the linear dependence relationships among the rows of a matrix.
*They preserve any relationship between columns, but not row; try a small example*

T — If the two $n \times n$ matrices $A, B$ are invertible, then so is $AB$ and $(AB)^{-1} = B^{-1}A^{-1}$
*Check:* $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I_n$, *similarly on the other side*

F — For a system $A\mathbf{x} = \mathbf{b}$ to be consistent, $\mathbf{b}$ must be in the row space of $A$.
$\mathbf{b}$ *must be in the column space.*

T — If every column of the coefficient matrix of a linear system $A\mathbf{x} = \mathbf{b}$ contains a pivot, then the system has at most one solution.
*No free variables.*

T — The map $T : \mathbf{R}^3 \to \mathbf{R}^3$ which rotates vectors by angle $\pi/4$ about the $z$-axis is linear.
*Geometry: it fixes the origin, preserves parallelograms, straight lines and distances, therefore it preserves linear combinations of vectors.*

T — Two matrices with the same column space must have the same left nullspace.
*The left nullspace is the set of linear equations that hold on the column space.*

T — If the coefficient matrix of a linear system $A\mathbf{x} = \mathbf{b}$ has a pivot in every row, then the system has at least one solution.
*You can always solve by choosing the values of the pivot variables appropriately.*

T — Each column of the product $AB$ is a linear combination of the columns of $A$, with weights taken from the corresponding column of $B$.
*A column of $AB$ is $A$ times the respective column of $B$; now interpret the matrix-vector product as stated.*

T — If $A$ is a $5 \times 5$ matrix and the system $A\mathbf{x} = \mathbf{b}_j$ is consistent for all $j = 1, \ldots, 5$, for some basis $\mathbf{b}_1, \ldots, \mathbf{b}_5$ of $\mathbf{R}^5$, then $A$ is invertible.
$\mathrm{Col}(A) = \mathbf{R}^5$ *so five pivots.*

F — For any $n \times n$ matrices $A, B$, we have $(AB)^T = A^T B^T$.
*Correct formula:* $(AB)^T = B^T A^T$.

**Question 2.** (18 points) The questions are worth 3 points each, if you select ALL correct answers (some questions have more than one correct answer). Partial credit is given, BUT an incorrect answer will forfeit all credit for the respective question. (It will not carry a negative score, though.)
**Please transfer your choices into the table below.**

| | | | | | | |
|---|---|---|---|---|---|---|
| #1 | | | | d | | |
| #2 | a | | | | | f |
| #3 | a | b | | | | |
| #4 | | b | | | | |
| #5 | | b | | | | |

| | | | | | | |
|---|---|---|---|---|---|---|
| #6 | | | c | d | | |
| #7 | | b | | d | | |
| #8 | | b | | d | | f |
| #9 | | | | d | | f |

1. For which real numbers $h$ is the following system consistent?
(Select the answer that contains all possible values.)

$$\begin{cases} x+y+z = & 1-h^2 \\ x-2y+3z = & 3 \\ 2x-y+4z = & h^2-4 \end{cases}$$

(a) For $h=1$ (b) For $h=\pm 1$ (c) For all values of $h$
(d) For $h=\pm 2$ (e) For $h=\pm\sqrt{2}$ (f) For no values of $h$

2. Which of the following matrices is in reduced echelon form? (Select all correct answers)

(a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3. Which of the following maps $T : \mathbf{R}^2 \to \mathbf{R}$ is NOT a linear transformation?
($(x, y)$ are the standard coordinates in $\mathbf{R}^2$.)

(a) $T(x,y) = x+y+1$ (b) $T(x,y) = x^2+y^2-(x+y)^2$ (c) $T(x,y) = \frac{x(x^2+y^2+1)}{x^2+y^2+1}$
(d) $T(x,y) = x+y$ (e) $T(x,y) = y-x$ (f) $T(x,y) = 2(x+2)-4(y+1)$

4. The dimensions of the row space and of the left nullspace of a matrix add up to

(a) The number of pivots (b) The number of rows (c) The number of free variables
(d) The number of solutions of the system (e) The number of columns (f) It depends on the pivot positions

5. In the basis $\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$ of $\mathbf{R}^2$, the coordinates of the vector $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$ are (in order)

(a) 1 and 1
(b) 2 and 1
(c) 4 and 5
(d) 1 and 2
(e) 2 and 2
(f) None of the above

6. Which of the following conditions ensures that a matrix $A$, of general shape, is invertible?

(a) Row-reduction finds a pivot on every row
(b) The reduced echelon form of $A$ contains only zeroes and ones
(c) There exists a matrix $B$ with $AB$, $BA$ both equal to an identity matrix
(d) Row-reduction finds a pivot on every row and every column
(e) $\mathrm{Nul}(A) = \{\mathbf{0}\}$
(f) None of the above

7. We are told that a linear transformation $L : \mathbf{R}^3 \to \mathbf{R}^3$ has 2-dimensional range.

In addition, $L(\mathbf{e}_1) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $L(\mathbf{e}_3) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Which of the following are possible values for $L(\mathbf{e}_2)$?

(a) $[3, 1, 1]^T$
(b) $[5, 3, 2]^T$
(c) $[2, 3, 1]^T$
(d) $[4, 3, 1]^T$
(e) $[4, 3, 2]^T$
(f) Any vector in $\mathbf{R}^3$

8. Under which of the circumstances below can we be certain that a system $A\mathbf{x} = \mathbf{b}$, with a $4 \times 5$ matrix $A$, has at least one solution?

(a) Always
(b) When $A$ has four pivots
(c) When $\mathbf{b}$ is in $\mathrm{Nul}(A)$
(d) When $\mathbf{b} = \mathbf{0}$
(e) When $A$ has a left nullspace
(f) When $\mathbf{b}$ is in $\mathrm{Col}(A)$

9. Choose the answers which only list well-defined multiplications, given the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}; \quad C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(Note: the answers will not contain all the possible well-defined multiplications.)

(a) $AB, AC, CA, CB$
(b) $AB, BA, A^TC, A^2$
(c) $AB, CA, B^TA, AC^T$
(d) $AB, BA^T, CA, C^TC$
(e) $B^2, B^TA, C^2, A^TB$
(f) $A^TA, AB, C^TA, BA^T$

**Question 3.** (15 points, 12+3)
(a) For the matrix $A$ below, find bases of the nullspace, column space, row space and left nullspace. Make sure your method is clear.
(b) For what value of $h$ does $[2,5,h]^T$ lie in the column space? Explain.

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 3 & 3 & 4 \\ 3 & 5 & 5 & 5 \end{bmatrix}$$

Answers: (a) The upper echelon form of $[A|I_3]$ is

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 5 & -3 & 2 & 0 \\ 0 & 1 & 1 & -2 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

Column space basis: $[1,2,3]^T, [2,3,5]^T$
Row space basis: $[1,0,0,5], [0,1,1,-2]$ (but you could also use the rows of the upper echelon form)
Nullspace basis: $[0,-1,1,0]^T, [-5,2,0,1]^T$
Left Nullspace basis: $[-1,-1,1]$
(b) $h = 7$ by testing against the left nullspace basis

**Question 4.** (12 points)
A linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ sends $[1,2]^T$ to $[2,4]^T$ and $[5,6]^T$ to $[6,8]^T$. Find the matrix of $T$ in the standard basis.
Express the standard basis vectors in terms of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\frac{3}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{5}{4}\mathbf{v}_1 - \frac{1}{4}\mathbf{v}_2$$

So $T(\mathbf{e}_1) = -\frac{3}{2}T(\mathbf{v}_1) + \frac{1}{2}T(\mathbf{v}_2) = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$, $T(\mathbf{e}_2) = \frac{5}{4}T(\mathbf{v}_1) - \frac{1}{4}T(\mathbf{v}_2) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and the standard matrix for $T$ is $\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$.

**Question 5.** (14 points)
For each of the following two matrices, find the inverse, or else explain why it is not invertible.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 10 \\ 3 & 2 & 7 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & \pi & 2 & 1 \\ 3 & 7 & 11 & 19 & 31 \end{bmatrix}. \quad A^{-1} = \frac{1}{2}\begin{bmatrix} -15 & 8 & -5 \\ -2 & 2 & -2 \\ 7 & -4 & 3 \end{bmatrix}$$

$B$ is not invertible because the first three rows are linearly dependent (the second is the average of the first and third), so the left nullspace is not zero and we will be missing at least one pivot.