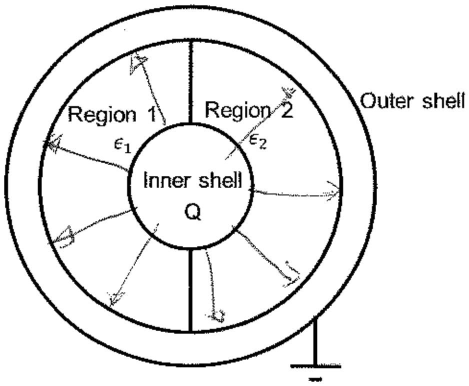
1. **(Gauss's Law and Boundary conditions)** We have two concentric **spherical shells** made of perfect **conductors**. Two different dielectric are filled in between the two shells, as shown below. On the left side the permittivity is ϵ_1, and on the right side permittivity is ϵ_2. The inner shell carries a total amount of charge Q, whereas the outer shell is grounded. Please find the electric field in region 1 and region 2.



– $\vec{E}$ is always in $\hat{r}$ direction.

– Boundary condition: $E_{1t} = E_{2t}$ between Regions 1 and 2.

$\hat{r}$ is in tangential direction.

– Therefore, $\vec{E}_1 = \vec{E}_2$.

Gauss' Law: $\oiint \epsilon \vec{E} \cdot d\vec{A} = Q$

– Area is spherical surface for each half.

– $\vec{E}$ is const over area and normal to surface.

$\rightarrow \epsilon_1 E \cdot 2\pi r^2 + \epsilon_2 E \, 2\pi r^2 = Q$

$$E = \frac{Q}{2\pi(\epsilon_1+\epsilon_2) r^2}$$

$$\boxed{\vec{E} = \frac{Q}{2\pi(\epsilon_1+\epsilon_2) r^2}\hat{r}}$$

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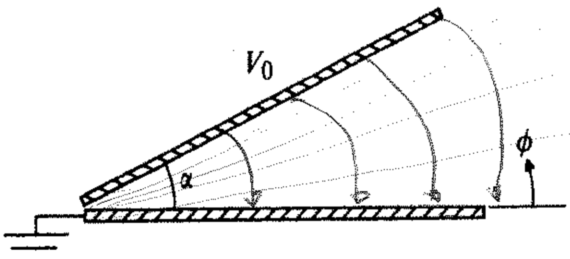
2. **(Laplace equation and Boundary conditions)** Two infinite conducting planes maintained at potentials 0 and V_0 form a wedge-shaped configuration, as shown in the figure. Determine the potential distributions for the regions: (a) $0 < \phi < \alpha$ (b) $\alpha < \phi < 2\pi$.



Since there is no charge in the free dielectric, we can use Laplace equation in cylindrical coordinates.

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

we see that the potential only varies by ϕ, as the conducting planes are infinite in r and z.

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

$$\frac{\partial^2 V}{\partial \phi^2} = 0 \rightarrow V(\phi) = c_1\phi + c_2$$

For $0 < \phi < \alpha$, we have boundary condition.

$V(0) = 0, \quad V(\alpha) = V_0$

$$\rightarrow \begin{matrix} c_1(0) + c_2 = 0 \\ c_1(\alpha) + c_2 = V_0 \end{matrix} \rightarrow \begin{matrix} c_2 = 0 \\ c_1 = \frac{V_0}{\alpha} \end{matrix} \rightarrow \boxed{V(\phi) = \frac{V_0}{\alpha}\phi, \quad 0 < \phi < \alpha}$$

For $\alpha < \phi < 2\pi$, we'll reparameterize $\theta = \phi - \alpha \quad 0 < \theta < 2\pi - \alpha$

Boundary conditions

$$V(\theta) = c_1(\theta) + c_2$$

$V(\phi = \alpha) = V(\theta = 0) = c_1(0) + c_2 = V_0$

$V(\phi = 2\pi) = V(\theta = 2\pi - \alpha) = c_1(2\pi - \alpha) + c_2 = 0$

$$\rightarrow \begin{matrix} c_2 = V_0 \\ c_1 = -\frac{V_0}{2\pi - \alpha} \end{matrix}$$

$$V(\theta) = \frac{-V_0}{2\pi - \alpha}\theta + V_0$$

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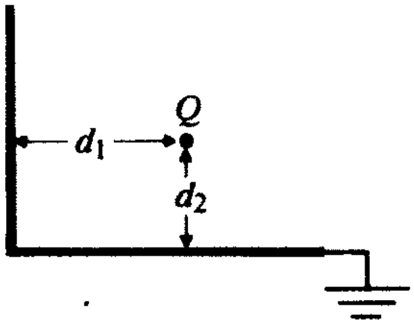
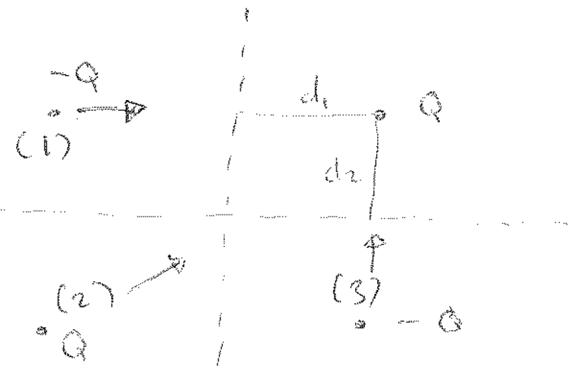
$$\rightarrow \boxed{V(\phi) = \frac{-V_0}{2\pi - \alpha}(\phi - \alpha) + V_0}$$

$\alpha < \phi < 2\pi$

3. **(Columb's law and image method)** A positive point charge Q is located at distance d_1 and d_2, respectively, from two grounded perpendicular conducting half-planes (infinitely long), as shown in the figure. **Determine the forces on Q caused by the charges induced on the planes.**





To make the equi-potential plane, we need image method twice.

$$\vec{F}_{(1)} = \frac{-Q \cdot Q}{4\pi\varepsilon_0 (2d_1)^2} \cdot \hat{x} = \frac{-Q^2}{16\pi\varepsilon_0 d_1^2}\hat{x}$$

$$\vec{F}_{(2)} = \frac{Q \cdot Q}{4\pi\varepsilon_0 (2d_1^2 + 2d_2^2)} \cdot \left(\frac{d_1\hat{x} + d_2\hat{y}}{\sqrt{d_1^2+d_2^2}} \right) = \frac{Q^2}{16\pi\varepsilon_0 (d_1^2+d_2^2)^{3/2}} \left(d_1\hat{x} + d_2\hat{y} \right)$$

$$\vec{F}_{(3)} = \frac{-Q \cdot Q}{4\pi\varepsilon_0 (2d_2^2)} \cdot \hat{y} = \frac{-Q^2}{16\pi\varepsilon_0 d_2^2}$$

$$\therefore \vec{F}_{net} = \frac{Q^2}{16\pi\varepsilon_0} \left(\hat{x}\left(\frac{d_1}{\sqrt{d_1^2+d_2^2}^{\,3}} - \frac{1}{d_1^2} \right) + \hat{y}\left(\frac{d_2}{\sqrt{d_1^2+d_2^2}^{\,3}} - \frac{1}{d_2^2} \right) \right)$$

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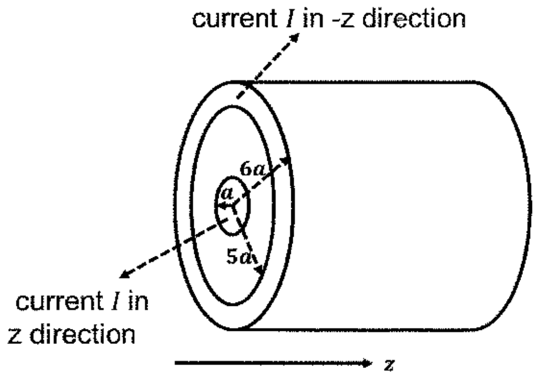
4. **(Magnetic potential and Laplace equation)** A coaxial transmission line shown below has an inner conductor with radius ***a*** and outer conductor between $r=5a$ and $r=6a$. The inner and outer conductors both carry current I in opposite directions, $\hat{z}$ and $-\hat{z}$, respectively. The magnetic potential A = 0 at $r = 5a$. **Find the vector magnetic potential** between two conductors, i.e. $\cancel{5}a<r<\cancel{6}a$. **(hint: A only has r dependence.)**

a 5a



By Ampere's law,

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$$

$$\vec{B} = \nabla \times \vec{A} = \frac{1}{r}\begin{vmatrix} \hat{r} & \hat{\phi} r & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \hat{r}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{z}\frac{1}{r}\left(\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right.$$

Since $\vec{A}$ only has r-dependency, $\partial/\partial\phi$ and $\partial/\partial z$ go to zero.

$$\hat{\phi}\left(-\frac{\partial A_z}{\partial r}\right) = \frac{\mu_0 I}{2\pi r}\hat{\phi} \longrightarrow A_z = -\frac{\mu_0 I}{2\pi}\ln|r| + c_1$$

$$\hat{z}\frac{\partial}{\partial r}(rA_\phi) = 0 \longrightarrow A_\phi = \frac{c_2}{r}$$

Since we set the boundary condition that $A(r=5a)=0$,

$$\frac{-\mu_0 I}{2\pi}\ln|5a| + c_1 = 0 \longrightarrow c_1 = \frac{\mu_0 I}{2\pi}\ln(5a)$$

$$\frac{c_2}{5a} = 0 \longrightarrow c_2 = 0$$

So now we have the complete solution:

$$\boxed{\vec{A} = \hat{z}\left(\frac{\mu_0 I}{2\pi}\left(\ln(5a) - \ln(r)\right)\right) = \hat{z}\left(\frac{\mu_0 I}{2\pi}\ln\left(\frac{5a}{r}\right)\right)}$$

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