Ald:

# MATH 185-S2 FALL 2008
# FINAL EXAM

Please write your name on each blue-book, and the number of blue-books used, if you use more than one. You have until 8:00pm. Write all proofs in *full sentences* and show your work whenever possible. There are four problems, skip ahead if you get stuck. Good luck!

(1) (24 pts)

(a) Define the notion of holomorphic function. Define the notion of integral of a continuous function along a smooth curve. State Goursat's Theorem.

(b) Define the notion of simply connected domain. State Cauchy's Theorem for a simply connected domain. Define the notion of Principal part of a function at a pole.

(c) State Riemann's Theorem on Removable Singularities. Define the notion of meromorphic function. State the Open Mapping Theorem.

(2) (24 pts) Label each of the following statements as True or False. Justify your answers.

(a) The set of all functions $f \in H(\mathbb{C})$ such that $|f(z)| \leq |\mathrm{Im}(z)|$ for all $z \in \mathbb{C}$, is infinite.

(b) All the roots of the equation $2z^5 - 6z^2 + z + 1 = 0$ lie in the disc $D_2(0)$.

(c) Let $\Omega \subset \mathbb{C}$ be open and let $z_0 \in \Omega$. Then a function $f \in H(\Omega \setminus \{z_0\})$ has a pole at $z_0$ if and only if $\mathrm{Res}_{z_0} f(z) \neq 0$.

(d) If $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$, then every conformal map $\mathbb{D}^* \to \mathbb{D}^*$ is a rotation.

(3) (26 pts)

Let $f$ be an entire function such that $|f(z^2)| \leq 2|f(z)|$ for all $z \in \mathbb{C}$, and let $M := \sup_{z \in \partial D_2(0)} |f(z)|$.

(a) Use induction to show that $|f(z^{2^n})| \leq 2^n |f(z)|$.

(b) Use part a) to show that if $|w| = 2^{2^n}$, then $|f(w)| \leq 2^n M$.

(c) Use Cauchy's Inequalities to show that for each integer $m \geq 1$, $|f^{(m)}(0)| \leq M(2^{n-2^n})$.

(d) Conclude that $f$ is a constant function.

(4) (26 pts)

Compute the following integrals, carefully justifying each step. (*Hint*: For part a), integrate along a circle. For part b), integrate along a rectangle in the upper half plane).

(a)

$$\int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2}, \qquad |a| < 1.$$

(b)

$$\int_{-\infty}^{\infty} \frac{\cos(tx)}{\cosh(x)} dx\,, \qquad t > 0\,.$$

(c) (EXTRA CREDIT, 10 pts) For each positive integer $n$, compute

$$\int_0^{\pi} (\sin\theta)^{2n} d\theta\,.$$