Math H185
Sarason

May 16, 2006

Final Examination

Open book, open notes. The points for each question are in parentheses.

1. (15) (a) For which positive numbers a will at least one value of i^a be real?

   (b) For which positive a will all values of i^a be real?

2. (15) Find the images under the linear-fractional transformation $\varphi(z) = \frac{z-i}{z+i}$ of the right half-plane Re $z > 0$, the left half-plane Re $z < 0$, and the sector $\frac{\pi}{4} < \text{Arg } z < \frac{3\pi}{4}$.

3. (10) Let the power series $\sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R, where $0 < R < \infty$. Let the power series $\sum_{n=0}^{\infty} b_n z^n$ have radius of convergence ∞. Prove the power series $\sum_{n=0}^{\infty} a_n b_n z^n$ has radius of convergence ∞.

4. (15) Evaluate

$$\int_{|z|=3\pi} \frac{z^n}{e^z - 1} dz, \qquad n = 0, 1, 2, \ldots,$$

   where the circle $|z| = 3\pi$ has the counterclockwise orientation.

5. (15) Let f be a complex-valued harmonic function in a domain G such that f^2 is also harmonic. Prove either f is holomorphic or $\bar{f}$ is holomorphic. (Suggestion: The complex differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ could be useful here.)

6. (15) The entire function f is said to be of exponential type if there are positive constants C and k such that $|f(z)| \leq Ce^{k|z|}$ for all z. Prove that if f is of exponential type then f' is also of exponential type.

7. (15) Let the function f be holomorphic in the strip $-1 < \text{Im } z < 1$, real on the real axis, and of positive imaginary part in the strip $0 < \text{Im } z < 1$.

   (a) Prove f has negative imaginary part in the strip $-1 < \text{Im } z < 0$.

   (b) Prove $f'(x) \geq 0$ for x real.

   (c) Prove $f'(x) > 0$ for x real.

8. (20) Evaluate

$$\int_0^{\infty} \frac{(\ln x)^2}{1+x^2} dx.$$

   Justify each step.

9. (10) Let f be an entire function such that the set $\mathbb{R} \cap f^{-1}(\mathbb{R})$ has a finite limit point. Prove $f(x)$ is real for all real x.