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MATH 185 FINAL

Do not write your answers on this sheet. Instead please write your name, your student ID, and all your answers in your blue books. Total: 100 pts., 2 hours and 50 minutes.

(1) (4 pts. each) For each of (a)-(e) below: If the proposition is true, write TRUE *and explain why it is true.* If the proposition is false, write FALSE *and give a counterexample.* (Please do not use the abbreviations T and F, since in handwriting they are sometimes indistiguishable.)

(a) If $f\colon \mathbb{C} \to \mathbb{C}$ is a function such that the functions $\operatorname{Re} f(x+iy)$ and $\operatorname{Im} f(x+iy)$ are differentiable (as functions $\mathbb{R}^2 \to \mathbb{R}$) at every point $(x,y) \in \mathbb{R}^2$, then f is an entire function.

(b) If $\ell(z)$ is a branch of $\log z$ on a domain G, and $z_1, z_2 \in G$ satisfy $z_1/z_2 \in G$, then $\ell(z_1/z_2) = \ell(z_1) - \ell(z_2)$.

(c) If γ is a closed curve contained in the set $\mathbb{C}^*$ of nonzero complex numbers, then

$$\int_\gamma \frac{1}{z^2}\, dz = 0.$$

(d) If a Laurent series centered at 0 converges at $4i$, then it converges also at -1.

(e) If $R > 0$, the domain $G_R := \{z \in \mathbb{C} : |z| > R\}$ is simply connected.

(2) (6 pts.) Let a and b be distinct complex numbers. Find, in terms of a and b, all complex numbers c such that a, b, c are the vertices of a triangle with a 30° angle at a, a 60° angle at b, and a 90° angle at c.

(3) (5 pts.) Sketch the set of $z \in \mathbb{C}$ such that $\operatorname{Re}(e^z) < 0$.

(4) (8 pts.) Does the series $\sum_{n=0}^{\infty} z^n/n!$ converge uniformly on $\mathbb{C}$? (Justify your answer.)

(5) (8 pts.) Let $f(z)$ and $g(z)$ be functions that are holomorphic on a neighborhood of 0, and assume that $g(z)$ has a zero of order 2 at $z = 0$. Let $a_0 = f(0)$, $a_1 = f'(0)$, $b_2 = g''(0)$, $b_3 = g'''(0)$. Find a formula for the residue of $f(z)/g(z)$ at $z = 0$ in terms of a_0, a_1, b_2, b_3.

(6) (8 pts.) Let $G = \{z \in \mathbb{C} : |z - 2i| > 1\}$. Let γ_0 be the straight line path from 4 to -4, and let γ_1 be the path $\gamma_1(t) = 4e^{it}$ for $t \in [0, \pi]$. Are γ_0 and γ_1 (fixed-point) homotopic in G? Use complex integration to prove your answer.

(more problems on back)

(7) (12 pts. each) Evaluate the following definite integrals:

(a) $\displaystyle\int_0^{2\pi} \frac{1}{5+3\sin\theta}\,d\theta$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(z^2+a^2)(z^2+b^2)}\,dz$ where $0<a<b$.

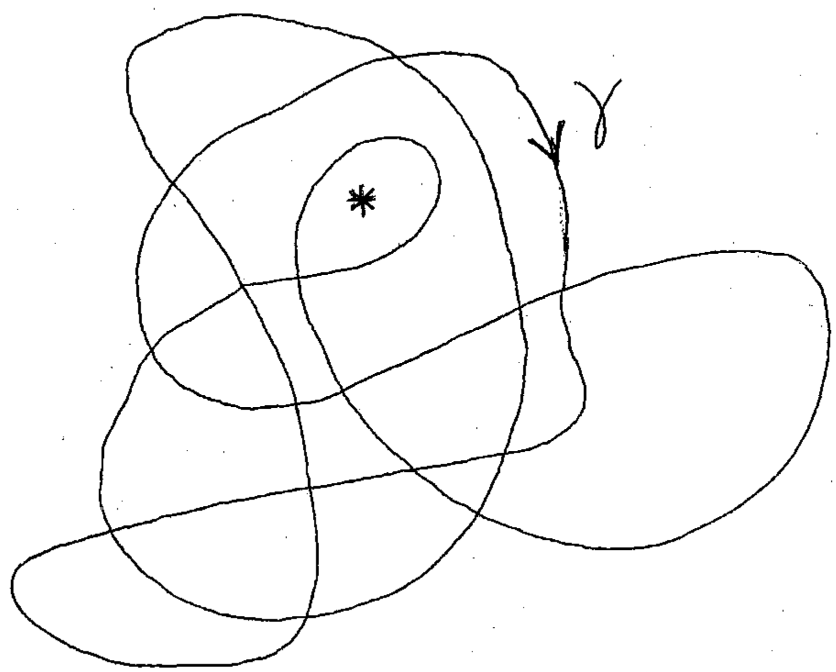
(8) (10 pts.) Let $f(z)$ be a function that is holomorphic in the region where $0<|z|<1$. Assume that $f(1/n)=0$ for all integers $n\geq 2$, but $f(z)$ is not identically zero. Prove that $f(z)$ has an essential singularity at $z=0$.

(9) (8 pts.) Let $n\geq 1$, and let $a_0, a_1, \ldots, a_n$ be complex numbers such that $a_n \neq 0$. For $\theta\in\mathbb{R}$, define

$$f(\theta)=a_0+a_1e^{i\theta}+a_2e^{2i\theta}+\cdots+a_ne^{ni\theta}.$$

Prove that there exists $\theta\in\mathbb{R}$ such that $|f(\theta)|>|a_0|$.

(10) (3 pts.) Let γ be the closed curve illustrated below, and let c be the complex number marked by the $*$. What is $\mathrm{ind}_\gamma(c)$?



This is the end! At this point, you may want to look over this sheet to make sure you have not omitted any problems. In particular, note that problem 1 has five parts, and problem 7 has two parts. Check that your answers make sense! Please take this sheet with you as you leave.