Prof. Bjorn Poonen
May 14, 2004, 8:10-11:00am, in 60 Evans Hall

# MATH 140 FINAL

*Do not write your answers on this sheet.* Instead please write your name, your student ID, and all your answers in your blue books. Total: 100 pts., 2 hours and 50 minutes.

**(1)** (5 pts. each) For each of (a)-(e) below: If the proposition is true, write TRUE *and explain why it is true.* If the proposition is false, write FALSE *and give a counterexample.* (Please do not use the abbreviations T and F, since in handwriting they are sometimes indistiguishable.)

(a) If $S$ is a regular surface in $\mathbb{R}^3$, and every $p \in S$ has a neighborhood in $S$ that is orientable, then $S$ is orientable.

(b) If $\phi\colon S \to \bar{S}$ is a conformal diffeomorphism between regular surfaces in $\mathbb{R}^3$, then $\phi$ is a local isometry.

(c) If every point on a nonempty connected compact surface $S$ in $\mathbb{R}^3$ is an elliptic point, then $S$ is homeomorphic to a sphere.

(d) Let $\pi\colon S^2 \to P^2$ be the usual map sending each $p$ on the unit sphere $S^2$ to the pair $\{p, -p\}$, which represents a single point on the projective plane $P^2$. Let $f\colon P^2 \to \mathbb{R}$ be a function. Then $f$ is differentiable if and only if the composition $f \circ \pi\colon S^2 \to \mathbb{R}$ is differentiable.

(e) If $\phi\colon S \to \bar{S}$ is an orientation-preserving isometry between oriented regular surfaces in $\mathbb{R}^3$, then the mean curvature of $S$ at a point $p$ equals the mean curvature of $\bar{S}$ at $\phi(p)$.

**(2)** (15 pts.) Let $C$ be a regular curve in $\mathbb{R}^3$ whose curvature vanishes nowhere. Let $\lambda$ be a positive real number, and let $\bar{C}$ be the image of $C$ under the map $v \mapsto \lambda v$. Suppose $p \in C$. Compute the torsion $\bar{\tau}$ of $\bar{C}$ at $\lambda p$ in terms of $\lambda$ and the torsion $\tau$ of $C$ at $p$.

**(3)** Define $\mathbf{x}\colon \mathbb{R}^2 \to \mathbb{R}^3$ by $\mathbf{x}(u, v) = (u\cos v, u\sin v, v)$.

(a) (15 pts.) Prove that the set $S = \mathbf{x}(\mathbb{R}^2)$ is a regular surface in $\mathbb{R}^3$.

(b) (15 pts.) Calculate the Gaussian curvature of $S$ at $\mathbf{x}(u, v)$, as a function of $u$ and $v$.

**(4)** (15 pts) Let $S$ and $\bar{S}$ be oriented regular surfaces in $\mathbb{R}^3$, and let $N\colon S \to \mathbb{R}^3$ and $\bar{N}\colon \bar{S} \to \mathbb{R}^3$ be the associated differentiable fields of unit normal vectors. Let $\alpha\colon I \to \mathbb{R}^3$ be a regular parametrized curve whose trace is contained in $S \cap \bar{S}$. Suppose moreover that $S$ and $\bar{S}$ are orthogonal along $\alpha$ (i.e., for each $t$, the vectors $N(\alpha(t))$ and $\bar{N}(\alpha(t))$ are orthogonal). Show that the geodesic curvature of $\alpha$ as a curve in $S$ equals the normal curvature of $\alpha$ as a curve in $\bar{S}$, up to a sign.

**(5)** (15 pts.) Let $S$ be a surface of constant Gaussian curvature $-1$. Let $R$ be a simple geodesic $n$-gon in $S$. (That is, $R$ is a subset of $S$ homeomorphic to a closed disk, and the boundary of $R$ is the union of $n$ geodesics.) Prove that the area of $R$ is strictly less than $(n-2)\pi$.

This is the end! At this point, you may want to look over this sheet to make sure you have not omitted any problems. Please take this sheet with you as you leave.