George M. Bergman   Fall 1995, Math 250A   Friday, 13 October 1995
7 Evans Hall   **Midterm Exam**   12:10-1:00 PM

1. (25 points) Prove that every group G (not necessarily finite) has a maximal abelian subgroup. (I.e., an abelian subgroup A maximal among all abelian subgroups of G; not necessarily among all subgroups of G.)

2. Let G be a group, and $X = Gx_0$, $Y = Gy_0$ finite transitive G-sets. Recall that G_{x_0} means the subgroup $\{g \in G \mid gx_0 = x_0\}$, and G_{y_0} is defined analogously.
(a) (15 points) Show that $\text{card}(X) = [G : G_{x_0}]$. (This is a result proved in the text; hence in proving it, you may not call on that result from the text. But you may call on any results proved before it.)

(b) (15 points) Let $X \times Y$ be made a G-set by defining $g(x, y) = (gx, gy)$. Describe $G_{(x_0, y_0)}$ in terms of G_{x_0} and G_{y_0} (proving your description), and show from this result that $[G : G_{(x_0, y_0)}]$ is divisible by both $[G : G_{x_0}]$ and $[G : G_{y_0}]$.
   (Don't expect to use (a) in proving (b); rather, (a) and (b) will both be used to prove (c). Note that if you prove something for G_{x_0}, you do not need to repeat the proof to assert the same result for G_{y_0}.)

(c) (15 points) Suppose $\text{card}(X) = m$ and $\text{card}(Y) = n$. Show that if m and n are relatively prime, then the G-set $X \times Y$ is transitive.

3. (a) (10 points) Let S be a set, $\mathbf{Z}S$ the free abelian group on S, and q the canonical map from S to $\mathbf{Z}S$ (in Lang's notation: $q(s) = 1 \cdot s$ for each $s \in S$). State the *universal property* characterizing the free abelian group $\mathbf{Z}S$. (If you are not sure you remember this correctly, but are more confident of universal properties in general categories, you might do (b) and (c) first and then come back to (a).)

(b) (10 points) If $\mathbf{C}$ is a category, define what it means for an object I to be an *initial object* (in Lang's language, a *universal repelling object*) of $\mathbf{C}$.

(c) (10 points) In the situation of (a), say how to make the set of pairs (A, f) such that A is an abelian group and $f : S \rightarrow A$ a set-map into a category, so that the universal property of A is equivalent to the statement that $(\mathbf{Z}S, q)$ is initial in this category. In making the set of pairs into a category, say *how* morphisms and their composition are defined, but *don't* verify that the axioms of a category hold, or even worry whether CAT1 does hold under your formulation. However, show in a sentence or so why the universal property implies the initial-object condition. (For brevity, you need not prove the converse.)