

P. Vojta
Fall 2005

Math 250a Final Exam

3 hours

1. (40 points) Define the following. You do not need to define terms used in those definitions.
 - (a). Covariant functor.
 - (b). Free group (not necessarily abelian).
 - (c). Factorial ring.
 - (d). Purely inseparable element.
2. (35 points) Show that all groups of order $588 = 2^2 \cdot 3 \cdot 7^2$ are solvable. In doing this, you may use the fact that all groups of order 12 are solvable.
3. (25 points) Show that an object representing a covariant functor (if it exists) is unique up to unique isomorphism.
(This was proved in class. In proving it here, you may use any results proved in class up to the point where this result was stated.)
4. (30 points) Let \mathfrak{m} be a maximal ideal in a commutative ring A . Show that \mathfrak{m} is a prime ideal.
(This was proved in the book, and also in class, but by a different method. In proving it here, you may use any results from group theory, any definitions in ring theory, and easy identities in ring theory such as $0x = 0$, but no other results from ring theory.)
5. (30 points) Show that any module M over a ring A is isomorphic to a direct limit of finitely generated A -modules.
6. (30 points) A field k is said to be **perfect** if it has characteristic 0, or if it has characteristic $p \neq 0$ and all elements of k have a p^{th} root in k .
Show that if k is a perfect field, then all algebraic extensions of k are separable.
7. (40 points)
 - (a). Show that the polynomial $X^5 - 2$ is irreducible over \mathbb{Q} .
 - (b). Let E be the splitting field of $X^5 - 2$ over \mathbb{Q} . Describe this field (e.g., generators, degree).
 - (c). Describe the Galois group $\text{Gal}(E/\mathbb{Q})$ as an abstract group, and give explicit generators of this group as a subgroup of \mathcal{S}_5 .