EECS 70 Discrete Mathematics and Probability Theory
Spring 2014 Anant Sahai

Midterm 1

Exam location: 1 Pimentel, front half: SIDs ending in 2 or 6

PRINT your student ID: ____

PRINT AND SIGN your name: ________________, ________________ ____________________
(last) (first) (signature)

PRINT your Unix account login: cs70-____________

PRINT your discussion section and GSI (the one you attend): ____

Name of the person to your left: ____

Name of the person to your right: ____

Name of someone in front of you: ____

Name of someone behind you: ____

Section 0: Pre-exam questions (3points)

1. **What other classes are you taking this term? (1 pt)**

2. **What activity do you really enjoy? Describe how it makes you feel. (2pts)**

Do not turn this page until the proctor tells you to do so. You can work on Section 0 above before time starts.

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Section 1: Straightforward questions (37 points)

3. **Its Own Inverse (5pts)**

   For $p > 1$, prove that $p-1$ is always its own multiplicative inverse in mod p arithmetic.

4. **Actually Invert it (10pts)**

   What is the multiplicative inverse of 21 in mod 31 arithmetic?

   (You should show that you've gotten the right answer and tell us how you got it, but you can follow whatever path you want to get the answer.)

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[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

5. Gotta match 'em all (12pts)

We are considering matching up Trainers with starter Pokémon. After the great crusade to liberate sentient beings from bondage, no longer can Pokémon be caught and enslaved. Now, it has to be a mutually satisfying voluntary partnership — the Pokémon preferences matter too.

Consider the following set of preferences:

Pokémon	Trainer		
A	2	3	1
B	1	3	2
C	3	2	1

Trainer	Pokémon		
1	A	B	C
2	C	A	B
3	B	C	A

a) **What are the Pokémon-optimal and Trainer-optimal stable pairings?**
(no justification necessary)

b) Insert another Trainer 4 and Pokémon D into the universe and incorporate them into a set of preferences consistent with those given above. *i.e.* The relative ordering between the ones above must still remain the same. (e.g. A must still prefer 2 over 3, etc.) **Do this in some way such that the propose-and-reject algorithm with Trainers proposing and Pokémon rejecting involves all the trainers being rejected at least once.** Run the algorithm to show that this indeed happens for your proposed preferences.

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6. Prove it by induction (10pts)

The j-th harmonic number is defined as

$$H_j = \sum_{i=1}^{j} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}.$$

So $H_1 = 1, H_2 = 1.5, \ldots$.

Use induction to prove that for any positive integer n,

$$\sum_{j=1}^{n} H_j = H_1 + H_2 + \cdots + H_n = (n+1)H_n - n.$$

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Section 2: True/False (30 points)

For the questions in this section, determine whether the statement is true or false. If true, prove the statement is true. If false, provide a counterexample demonstrating that it is false.

7. Implications (10pts)

If $P \implies Q$, then $Q \implies P$.

Mark one: TRUE or FALSE.

8. Factorial and Mod (20pts)

If p is prime, then $(p-1) \equiv (p-1)! \mod p$.

Mark one: TRUE or FALSE.

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Section 3: Free-form Problems (60 points)

9. Make your own EGCD (20pts)

The extended GCD algorithm discussed in the notes leveraged the fact that $GCD(x,y) = GCD(y, x \bmod y)$.

Make an EGCD algorithm based instead on the fact that if $y \leq x$**, then** $GCD(x,y) = GCD(y, x-y)$**. Prove that your algorithm always terminates when given any natural numbers** (x,y) **and correctly returns a triple of integers** (d,a,b) **so that** $d = GCD(x,y)$ **and** $d = ax + by$**.**

Your algorithm should be recursive and you are not allowed to use the mod operation nor division nor multiplication in the algorithm. (You can use whatever you want in your proof.)

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10. Don't let it (this problem) go (20pts)

In Arendelle (the name of a fictional country), there are only two types of coins: one worth 31 cents and the other one worth 21 cents. Elsa used coins to buy a gift for Anna and paid 1,010 cents exactly. **How many 31-cent coins and 21-cent coins did Elsa use?**

(Think about what you have learned that can be applied and write down what you try. We also need to be able to understand your work so please give us comments to go with your calculations.)

(First HINT: Coins as cups.
Second HINT: Zero plus anything is …
Third HINT: Feel free to use other problems on this exam to help you.
Fourth HINT: Feel free to ignore these hints if they are not helpful.)

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11. (Optional) Asking makes the heart grow fonder (20pts)

Among the trolls, men and women have quantitative preferences for each other. So each man has a personal "liking" score for each woman that expresses how much he likes her and vice-versa, each woman has a "liking" score for each man. A higher score is better. Men and women start with some personalized liking-score lists for each other. Initial liking scores are all distinct positive real numbers.

What makes this land interesting is that the scores can change during the courtship process, but become frozen forever once everyone is married. In this land, they follow the male-propose, female-reject method of matching. (i.e. At every day, each troll man proposes to the troll woman he likes best out of all those who haven't rejected him yet while each woman says "maybe" to the current suitor she likes best and rejects the others proposing to her. When nobody is rejected, all the current "maybes" get married.)

Rogue couples are defined to be any whose **final** liking scores for each other are higher than their **final** scores for their spouses. Stable pairings have no rogue couples.

Suppose that each day a man proposes to a woman, his score for her increases by 5 points. Whenever a woman says "maybe" to a man, her score for him increases by 5 points but only if he had any current competition for her hand. (She gives no bonus points for being the only guy who shows up.) Their response to rejections is even more different. When a woman rejects a man, her score for him drops by a factor of 2 but his score for her increases by 6 points. (He wants more who he can't have.)

State and prove an appropriate "improvement lemma" for the troll women.

(HINT: do some examples that correspond to "shallowland" with tightly clustered initial scores.)

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[Doodle page! Draw us something if you want or give us suggestions or complaints.]